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Higher Koszul duality and connections with n-hereditary algebras

Johanne Haugland

Joint work with Mads H. Sandøy

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Introduction

Joint with Mads H. Sandøy: *Higher Koszul duality and connections with n-hereditary algebras* (2021, arXiv:2101.12743)

Introduction

Koszul duality ~~~~ **Higher homological algebra**

[BGS '96]

generalized Koszul algs.

[GRS '02]
[M '11]
} T-Koszul

[I '07...]

n-hereditary algs.

[IO '11], [IO '13], [HIO '14]

Plan of talk

1. n-hereditary algebras
2. Classical Koszul duality
3. Motivation
4. Higher Koszul duality and answer to motivating question
5. More results connecting higher Koszul duality and n-representation infinite algebras
6. (Higher almost Koszulity and n-representation finite algebras)

Conventions and notation

1. $n =$ positive integer
2. $k =$ algebraically closed field
3. All algebras are algebras over k
4. $D(-) = \operatorname{Hom}_k(-, k)$
5. A and $B =$ ungraded algebras
6. $\Lambda =$ positively graded algebra
7. $\operatorname{mod} A =$ finitely presented right A-modules
8. $\operatorname{gr} \Lambda =$ finitely presented graded right Λ-modules

n-hereditary algebras

Classical case ($n = 1$)

hereditary alg. A — $\operatorname{gl.dim} A \leq 1$, $A \simeq kQ$

rep. finite $\iff$ $\forall$ indec. $P \in \operatorname{proj} A$ $\exists\, i \geq 0$ s.t. $\tau^{-i}P$ indec. inj.

rep. infinite $\iff$ $\nu_1^{-i} A$ conc. in degree 0 for $i \geq 0$

Check: $M \in \operatorname{mod} A$ indec. non-inj. $\Rightarrow$ $\nu_1^{-1} M$ conc. in degree 0

n-hereditary algebras

From now, let A be a finite dimensional algebra with $\operatorname{gl.dim} A \leq n$.

Nakayama functor

$$\nu = D\,\mathrm{RHom}_A(-, A)\colon\ \mathcal{D}^b(\operatorname{mod} A) \xrightarrow{\simeq} \mathcal{D}^b(\operatorname{mod} A)$$
$$\nu^{-1} = \mathrm{RHom}_A(DA, -)\colon\ \mathcal{D}^b(\operatorname{mod} A) \xrightarrow{\simeq} \mathcal{D}^b(\operatorname{mod} A)$$

We use the notation $\nu_n = \nu \circ [-n]$.

Auslander–Reiten translation

For $n = 1$, we have $\tau \simeq \mathrm{H}^0(\nu_1)\colon\ \operatorname{mod} A \to \operatorname{mod} A$

Higher Auslander–Reiten translation

$$\tau_n = \mathrm{H}^0(\nu_n)\colon\ \operatorname{mod} A \to \operatorname{mod} A$$

n-hereditary algebras

Definition

1. A is called *n-representation finite* if there for each indecomposable $P \in \operatorname{proj} A$ exists an integer $i \geq 0$ such that $\nu_n^{-i} P$ is indecomposable injective.
2. A is called *n-representation infinite* if $\mathrm{H}^i(\nu_n^{-j} A) = 0$ for $i \neq 0$ and $j \geq 0$.
3. A is called *n-hereditary* if it is either n-representation finite or n-representation infinite.

n-hereditary algebras

Classes of examples of n-representation finite algebras

1. Higher type A algebras [IO ’11]
2. Nakayama algebras with homogeneous relations [DI ’20, V ’19]
3. Iterated n-APR tilts of higher representation finite algebras [IO ’11]
4. Tensor products of ℓ-homogeneous higher representation finite algebras [HI ’11]

n-hereditary algebras

Classes of examples of *n*-representation infinite algebras

1. Higher type $\widetilde{A}$ algebras [HIO ’14]
2. Tensor products of higher representation-infinite algebras [HIO ’14]

n-hereditary algebras

Higher preprojective algebras

Given an n-hereditary algebra A, the $(n+1)$-*preprojective algebra* of A is given by

$$\Pi_{n+1}A = \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathcal{D}^b(A)}(A, \nu_n^{-i}A).$$

Koszul algebras

A graded algebra $\Lambda = \oplus_{i \geq 0} \Lambda_i$ which is generated in degrees 0 and 1 with Λ_0 semisimple is known as a *Koszul algebra* if

$$\operatorname{Ext}^i_{\operatorname{gr}\Lambda}(\Lambda_0, \Lambda_0\langle j \rangle) = 0$$

for $i \neq j$.

The *Koszul dual* of Λ is defined as

$$\Lambda^! = \bigoplus_{i \geq 0} \operatorname{Ext}^i_{\operatorname{gr}\Lambda}(\Lambda_0, \Lambda_0\langle i \rangle).$$

Koszul duality

Let Λ be a Koszul algebra and $\Lambda^!$ its Koszul dual. Given certain finiteness conditions, we have

$$\mathcal{D}^b(\operatorname{gr}\Lambda) \xrightarrow{\simeq} \mathcal{D}^b(\operatorname{gr}\Lambda^!).$$

Aim

Generalize the notion of Koszul algebras and get a *higher* version of the Koszul duality equivalence above.

Trivial extensions

Let A be a finite dimensional algebra. The *trivial extension of* A is

$$\Delta A = A \oplus DA$$

with multiplication $(a, f) \cdot (b, g) = (ab, ag + fb)$ for $a, b \in A$ and $f, g \in DA$.

ΔA can be graded with A in degree 0 and DA in degree 1.

Graded symmetric algebras

A finite dimensional algebra $\Lambda = \oplus_{i \geq 0} \Lambda_i$ is *graded symmetric* if $D\Lambda \simeq \Lambda\langle -a \rangle$ as graded Λ-bimodules for some integer a.

Note

1. Any graded symmetric algebra is self-injective.
2. The integer a must be equal to the highest degree of Λ.

Example

The trivial extension ΔA of a finite dimensional algebra A is graded symmetric.

Motivation

Finiteness condition

The category $\operatorname{gr}\Lambda$ is abelian if and only if Λ is *graded right coherent*, i.e. if every finitely generated homogeneous right ideal is finitely presented.

Some known equivalences

Let A be an n-representation infinite algebra with $\Pi_{n+1}A$ graded right coherent. We then have

$$\underline{\operatorname{gr}}\,\Delta A \simeq \mathcal{D}^b(\operatorname{mod} A) \simeq \mathcal{D}^b(\operatorname{qgr}\Pi_{n+1}A).$$

Happel   Minamoto, Mori

Motivating question

Does the equivalence $\underline{\operatorname{gr}}\,\Delta A \simeq \mathcal{D}^b(\operatorname{qgr}\Pi_{n+1}A)$ remind us of something?

Koszul duality and the BGG-correspondence

Let Λ be a Koszul algebra which is graded symmetric.

$$\begin{array}{ccc}
\mathcal{D}^b(\operatorname{gr}\Lambda) & \xrightarrow{\simeq} & \mathcal{D}^b(\operatorname{gr}\Lambda^!) \\
\downarrow & & \downarrow \\
\underline{\operatorname{gr}}\,\Lambda & \stackrel{\simeq}{\dashrightarrow} & \mathcal{D}^b(\operatorname{qgr}\Lambda^!)
\end{array}$$

$D^b(\operatorname{gr}\Lambda) / D^{perf}(\operatorname{gr}\Lambda) \xrightarrow{\simeq} \underline{\operatorname{gr}}\,\Lambda$

Rickard

BGG-corresp.

Koszul duality and the BGG-correspondence

Let Λ be a Koszul algebra which is graded symmetric.

$$\begin{array}{ccc} \mathcal{D}^b(\operatorname{gr}\Lambda) & \xrightarrow{\simeq} & \mathcal{D}^b(\operatorname{gr}\Lambda^!) \\ \downarrow & & \downarrow \\ \underline{\operatorname{gr}}\,\Lambda & \overset{\simeq}{\dashrightarrow} & \mathcal{D}^b(\operatorname{qgr}\Lambda^!) \end{array}$$

Motivating question

Is the equivalence $\underline{\operatorname{gr}}\,\Delta A \simeq \mathcal{D}^b(\operatorname{qgr}\Pi_{n+1}A)$ a consequence of some higher Koszul duality?

Generalized Koszul algebras

Tilting modules

Let A be a finite dimensional algebra. A finitely generated A-module T is called a *tilting module* if the following conditions hold:

1. $\text{proj.dim}_A T < \infty$;
2. $\text{Ext}^i_A(T, T) = 0$ for $i > 0$;
3. There is an exact sequence

$$0 \to A \to T^0 \to T^1 \to \cdots \to T^l \to 0$$

   with $T^i \in \text{add}\, T$ for $i = 0, \ldots, l$.

Generalized Koszul algebras

Let $\Lambda = \oplus_{i \geq 0} \Lambda_i$ be a positively graded algebra.

Definition

Let T be a finitely generated basic graded Λ-module concentrated in degree 0. We say that T is *graded n-self-orthogonal* if

$$\mathrm{Ext}^i_{\mathrm{gr}\,\Lambda}(T, T\langle j\rangle) = 0$$

for $i \neq nj$.

Generalized Koszul algebras

Definition

Assume $\operatorname{gl.dim} \Lambda_0 < \infty$ and let T be a graded Λ-module concentrated in degree 0. We say that Λ is *n-T-Koszul* or *n-Koszul with respect to T* if the following conditions hold:

1. T is a tilting Λ_0-module.
2. T is graded n-self-orthogonal as a Λ-module.

Generalized Koszul algebras

Definition

Assume $\mathrm{gl.dim}\,\Lambda_0 < \infty$ and let T be a graded Λ-module concentrated in degree 0. We say that Λ is *n-T-Koszul* or *n-Koszul with respect to T* if the following conditions hold:

1. T is a tilting Λ_0-module.
2. T is graded n-self-orthogonal as a Λ-module.

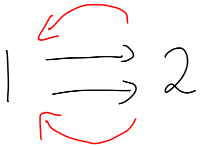
Definition

Let Λ be an n-T-Koszul algebra. The *n-T-Koszul dual of* Λ is given by

$$\Lambda^! = \oplus_{i \geq 0} \mathrm{Ext}^{ni}_{\mathrm{gr}\,\Lambda}(T, T\langle i \rangle).$$

Generalized Koszul algebras

Example



$\rightsquigarrow A$ 1-RI

$\rightsquigarrow \Delta A$

Check: $\mathrm{Ext}^i_{gr\,\Delta A}(A, A\langle j \rangle) = 0$ for $i \neq 2j$

$\rightsquigarrow \Delta A$ is 2-Koszul wrt. A

Generalized Koszul algebras

Proposition

Let A be an n-representation infinite algebra. The following statements hold:

1. The trivial extension ΔA is $(n+1)$-Koszul with respect to A.
2. We have $(\Delta A)^! \simeq \Pi_{n+1} A$ as graded algebras.

Higher Koszul duality

Theorem

Let Λ be a finite dimensional n-T-Koszul algebra and assume that $\Lambda^!$ is graded right coherent and has finite global dimension. Then there is an equivalence

$$\mathcal{D}^b(\operatorname{gr}\Lambda) \xrightarrow{\simeq} \mathcal{D}^b(\operatorname{gr}\Lambda^!)$$

of triangulated categories.

Higher Koszul duality and BGG-correspondence

Proposition

In the case where our algebra Λ is graded symmetric, the higher Koszul duality equivalence descends to yield an analogue of the BGG-correspondence

$$\begin{array}{ccc} \mathcal{D}^b(\operatorname{gr}\Lambda) & \xrightarrow{\simeq} & \mathcal{D}^b(\operatorname{gr}\Lambda^!) \\ \downarrow & & \downarrow \\ \underline{\operatorname{gr}}\,\Lambda & \overset{\simeq}{\dashrightarrow} & \mathcal{D}^b(\operatorname{qgr}\Lambda^!). \end{array}$$

Back to our motivating question

Corollary

Let A be an n-representation infinite algebra with $\Pi_{n+1}A$ graded right coherent. We then have:

$$\begin{array}{ccc} \mathcal{D}^b(\operatorname{gr} \Delta A) & \xrightarrow{\simeq} & \mathcal{D}^b(\operatorname{gr} \Pi_{n+1} A) \\ \downarrow & & \downarrow \\ \underline{\operatorname{gr}} \Delta A & \stackrel{\simeq}{\dashrightarrow} & \mathcal{D}^b(\operatorname{qgr} \Pi_{n+1} A) \end{array}$$

In particular, this holds whenever an n-representation infinite algebra A is *n-representation tame* as defined in [HIO '14].

A characterization

Tilting object

Let $\mathcal{T}$ be a triangulated category. An object T in $\mathcal{T}$ is a *tilting object* if the following conditions hold:

1. $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for $i \neq 0$;
2. $\mathrm{Thick}_{\mathcal{T}}(T) = \mathcal{T}$.

A characterization

Notation and standing assumptions

1. $\Lambda =$ graded symmetric algebra of highest degree $a \geq 1$
2. $\operatorname{gl.dim} \Lambda_0 < \infty$
3. $T \in \operatorname{gr} \Lambda$ satisfies:
   i) T is concentrated in degree 0
   ii) T is a tilting module over Λ_0
4. $\widetilde{T} = \bigoplus_{i=0}^{a-1} \Omega^{-ni} T\langle i \rangle$
5. $B = \operatorname{End}_{\underline{\operatorname{gr}} \Lambda}(\widetilde{T})$

A characterization

Theorem

The following statements are equivalent:

1. Λ is n-T-Koszul.
2. $\widetilde{T}$ is a tilting object in $\underline{\mathrm{gr}}\,\Lambda$ and B is $(na-1)$-representation infinite.

Corollary: A n-RI $\iff$ ΔA (n+1)-Koszul wrt. A

A characterization

Corollary

There is a bijective correspondence

$$\left\{\begin{array}{l}\text{isomorphism classes}\\ \text{of } n\text{-representation}\\ \text{infinite algebras}\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\text{isomorphism classes of graded symmetric}\\ \text{finite dimensional algebras of highest}\\ \text{degree } 1 \text{ which are } (n+1)\text{-Koszul with}\\ \text{respect to their degree } 0 \text{ part}\end{array}\right\}.$$

$$A \longmapsto \Delta A$$

$$\Lambda_0 \longleftarrow\!\shortmid \Lambda$$

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Thank you for your attention

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