Jordan-Hölder exact categories

Representation Theory and Related Topics Seminar
at Northeastern University

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PLAN

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2. Motivation
3. Exact categories
4. Jordan-Hölder exact categories
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8. Artin-Wedderburn exact categories
9. The case of module categories over Nakayama algebras
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THE ORIGINS OF RELATIVE HOMOLOGY

History of relative homology

- 1934 *Baer* introduced Ext for abelian groups
- 1940 *Baer* defines the Baer sum
- 1954 *Yoneda* proves *the classification theorem*, a one-to-one correspondence between the equivalence classes of the $n-$fold extensions of B by A and the elements of the abelian group ${Ext_{\Lambda}}^{n}(A, B)$
- 1955 *Buchsbaum* proves the existence of *Ext* for an exact category having enough projectives or enough injectives
- 1956 *Cartan and Eilenberg* generalize the notion extension groups
- 1957 *Buchsbaum* defines the extension functor *Ext without* using the projective and the injective objects
- 1958 *Hochschild* discusses the analogous of the *Ext* but applicable to a module theory that is *relativized* with respect to a given subring of the basic ring of operators
- 57-58 *Harrison* and *Heller* discuss similar problems, which make it natural to consider the extension functor on a specific exact categories

History of relative homology

The idea of *relative homological algebra for abstract categories* is about the selection of a class of extensions or, equivalently, a class of monomorphisms and epimorphisms.

History of relative homology

1961 *Butler and Horrocks* study relative homological algebras, but only for abelian categories.

They study how the derived functors behave under reduction of the exact structure

History of relative homology

Recent works:

1993 *Auslander and Ø.Solberg* discuss applying relative homological algebras to representation theory

1999 *Dräxler, Reiten, Smalø, Ø.Solberg + Keller* study the correspondence between exact structures and closed additive bifunctors of *Ext*

2005 *Auslander and Ø.Solberg* develope a general theory of *relative* cotilting modules for artin algebras

Motivation

WHY
DO WE WANT TO STUDY THIS SUBJECT ?

Motivation

Nice length function

Jordan-Hölder length improves [BHLR 18'] $\mathcal{E}-$length function

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Intersection and sum of subobjects

Jacobson radical, trace of subcategories, lattice of subobjects,...

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It leads to new characterisations of the important and popular quasi-abelian (functional analysis) and abelian categories

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Applications

The work by [Berktas, Crivei, Kaynarca, Keskin, Tütñcü, 21'] which generalize the theorems of uniqueness of uniform decompositions in abelian categories

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Jordan-Hölder property

[Enomoto 19'] $(\mathcal{A}, \mathcal{E})$ satisfies (JHP) if and only if the $\mathcal{E}-$Grenthendieck group is free

AXIOMATIC DEFINITION

Quillen's exact categories

Definition

An ***exact category*** is a pair $(\mathcal{A}, \mathcal{E})$ consisting of an additive category $\mathcal{A}$ and an exact structure $\mathcal{E}$ on $\mathcal{A}$.

Quillen's exact categories

Definition

An ***exact structure*** $\mathcal{E}$ on is a class of kernel-cokernel pairs (i, d) in which is closed under isomorphisms and satisfies the following axioms:

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(A1) the class of admissible monics (resp. admissible epics) is closed under composition

Quillen's exact categories

Definition

(A2) The push-out of an admissible monic i along an arbitrary morphism a exists and yields an admissible monic s_C:

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & B \\ {\scriptstyle a}\downarrow & \text{PO} & \downarrow{\scriptstyle s_B} \\ C & \underset{s_C}{\rightarrowtail} & S \end{array}$$

(A2)' The pull-back of an admissible epic h along a exists and yields an admissible epic p_B:

$$\begin{array}{ccc} P & \overset{p_B}{\twoheadrightarrow} & B \\ {\scriptstyle P_A}\downarrow & \text{PB} & \downarrow{\scriptstyle a} \\ A & \overset{h}{\twoheadrightarrow} & C. \end{array}$$

Extriangulated categories

Remark

$(\mathcal{A}, Ext^1{}_{\mathcal{E}}, \mathbb{I})$ is a Nakaoka-Palu *Extriangulated* category.

Theorems [BBGH, 7.34][BHLR, 5.4]:

Let $\mathcal{A}$ be an additive category. The map $\Phi : \mathcal{E} \mapsto Ext^1_{\mathcal{E}}(-,-)$ induces a *lattice isomorphism* between $(Ex(\mathcal{A}), \subseteq, \cap, \vee_{Ex})$ and $(Cbf(\mathcal{A}), \leq, \wedge, \vee_{Cbf})$.

Examples of exact structures

Smallest example

The minimal exact structure formed by all split short exact sequences.

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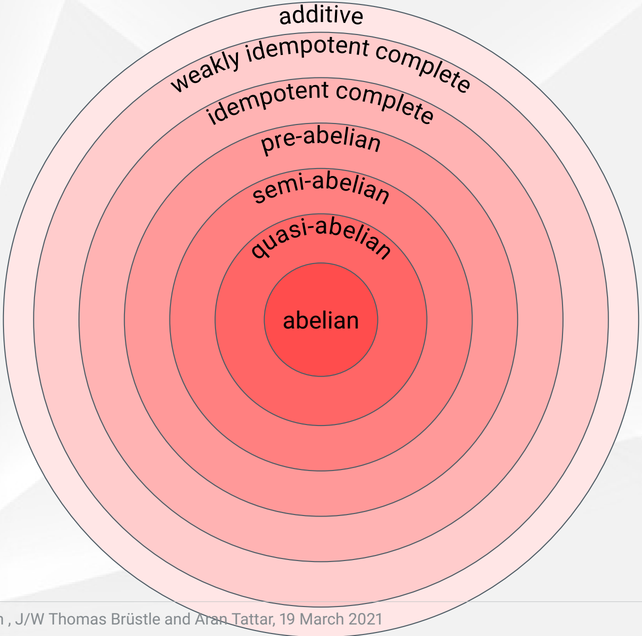
The minimal exact structure formed by all split short exact sequences.

[Largest example, Rump 2011]

There exists a unique maximal exact structure for any additive category $\mathcal{A}$.

Additive categories



[RW77, Definition]:

A kernel (A, f) is called *semi-stable* if for every push-out square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle t}\downarrow & \text{PO} & \downarrow{\scriptstyle s_B} \\ C & \xrightarrow[s_C]{} & S \end{array}$$

the morphism s_C is also a kernel. We define dually a *semi-stable* cokernel. A short exact sequence $A \rightarrowtail^{i} B \twoheadrightarrow^{d} C$ is said to be *stable* if i is a semi-stable kernel and d is a semi-stable cokernel. We denote by $_{sta}$ the class of all *stable* short exact sequences.

Definition

An additive category is *quasi-abelian* if it is *pre-abelian* and all kernels and cokernels are *semi-stable*.

Example

The maximal exact structure on a quasi-abelian category $\mathcal{A}$ consists of all short exact sequences on $\mathcal{A}$:

$$\mathcal{E}_{max} = \mathcal{E}_{all} = \mathcal{E}_{sta}.$$

JORDAN-HÖLDER PROPERTY

The Jordan-Hölder-Schreier theorem

Theorem

If an $A-$module X admits two composition series

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

and

$$0 = X'_0 \subset X'_1 \subset \cdots \subset X'_{m-1} \subset X'_m = X$$

then they are equivalent: $n = m$ and there exists a permutation σ of $\{0, 1, ..., n-1\}$ such that $X_{i+1}/X_i \cong X'_{\sigma(i)+1}/X'_{\sigma(i)}$.

Notations

—> Generalisation	
Abelian categories	Exact categories
subobjects $A \subseteq B$	$\mathcal{E}-$subobjects $A \rightarrowtail B$
simple subobjects $0 \subset S$	$\mathcal{E}-$simple subobject $0 \rightarrowtail S$
Composition series	$\mathcal{E}-$composition series
Jordan-Hölder property	$\mathcal{E}-$Jordan-Hölder property
Intersection, sum and Jacobson radical	New general intersection, sum and $\mathcal{E}-$radical
Artin-Wedderburn categories	$\mathcal{E}-$Artin-Wedderburn categories
Lattice of subobject	Poset of subobjects

$\mathcal{E}-$Jordan-Hölder property

Definition

Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A finite $\mathcal{E}-$composition series for an object X of $\mathcal{A}$ is a sequence

$$0 = X_0 \overset{i_0}{\rightarrowtail} X_1 \overset{i_1}{\rightarrowtail} \dots \overset{i_{n-2}}{\rightarrowtail} X_{n-1} \overset{i_{n-1}}{\rightarrowtail} X_n = X$$

where all i are *proper admissible monics* with $\mathcal{E}-$simple cokernel.

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where all i are *proper admissible monics* with $\mathcal{E}-$simple cokernel. We say an $(\mathcal{A}, \mathcal{E})$ is a *Jordan-Hölder exact category* if any two finite $\mathcal{E}$-composition series of X are equivalent.

Counter-example

C onsider $(\mathcal{A}_S, \mathcal{E}_{min})$ with $\mathcal{A}_S = \{v \in Vec_k | dim_k(v) \in S = \mathbb{N} \backslash \{1\}\}$.

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There are two non-equivalent $\mathcal{E}_{min}-$composition series for the object $X = k^6$:

$$0 \longrightarrow K^2 \rightarrowtail K^4 \rightarrowtail K^6$$

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$$0 \longrightarrow K^2 \rightarrowtail K^4 \rightarrowtail K^6 \qquad \ell_{\mathcal{E}_{min}}(K^6) = 2$$

$$0 \longrightarrow K^3 \rightarrowtail K^6. \qquad \ell_{\mathcal{E}_{min}}(K^6) = 1$$

Examples and Counter-examples

C onsider $\mathcal{R}ep\mathcal{Q}$ with $\mathcal{Q}$: $1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$

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C onsider $\mathcal{R}ep\mathcal{Q}$ with $\mathcal{Q}$: $1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$

The *A-R* sequences in $\mathcal{R}ep\mathcal{Q}$ are

(1) $S_2 \to P_1 \oplus P_3 \to I_2$

(2) $P_3 \to I_2 \to S_1$

(3) $P_1 \to I_2 \to S_3$.

Examples and Counter-examples

T he exact structures accordingly are
$\mathcal{E}_{min}, \mathcal{E}(1), \mathcal{E}(2), \mathcal{E}(3), \mathcal{E}(1,2), \mathcal{E}(1,3), \mathcal{E}(2,3), \mathcal{E}_{max}$.

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$(\mathcal{R}ep\mathcal{Q}, \mathcal{E}(1))$ is not Jordan-Hölder since there are non-equivalent $\mathcal{E}(1)-$composition series $0 \to S_2 \to P_1 \oplus P_3$ and $0 \to P_1 \to P_1 \oplus P_3$.

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$(\mathcal{R}ep\mathcal{Q}, \mathcal{E}(2,3))$ is not Jordan-Hölder since there are non-equivalent $\mathcal{E}(2,3)-$composition series $0 \to P_1 \to I_2$ and $0 \to P_3 \to I_2$.

Intersection and sum

OUR APPROACH:
STUDY THE INTERSECTION AND SUM OF SUBOBJECTS

We call **A.I** the exact categories admitting **A**dmissible **I**ntersections:

[HR, Definition 4.3]:

An exact category $(\mathcal{A}, \mathcal{E})$ is called an ***AI**-category* if $\mathcal{A}$ is pre-abelian satisfying:

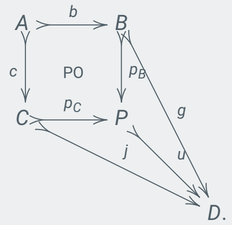
(*AI*) The pull-back (P, p_A, p_B) of two admissible monics *a* and *b* exists and yields two admissible monics:

$$\begin{array}{ccc} P & \overset{p_B}{\rightarrowtail} & B \\ {\scriptstyle p_A}\downarrow & \text{PB} & \downarrow{\scriptstyle b} \\ A & \underset{a}{\rightarrowtail} & C. \end{array}$$

[HR, Definition 4.4]:

An exact category $(\mathcal{A}, \mathcal{E})$ is called an ***AS-category*** if:

(AS) The morphism u, given by the universal property of the push-out (P, p_B, p_C) is an admissible monic.

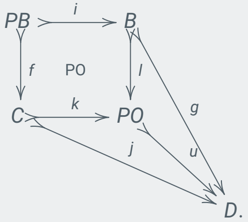


(AIS) categories

We call **A.I.S-categories** the exact categories admitting **A**dmissible **I**ntersections and **S**ums:

[HR, Definition 4.5]:

An exact category $(\mathcal{A}, \mathcal{E})$ is an ***AIS****-category* if it is an *AI*-category and moreover, the push-out along these pull-backs yields an admissible monic *u*:



Quasi-abelian categories

[BHT, Theorem 4.12]:

Every AI-category $\mathcal{A}$ is quasi-abelian.

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Every AI-category $\mathcal{A}$ is quasi-abelian.

[BHT, Theorem 4.17][HSW, Theorem 6.1]:

A category $(\mathcal{A}, \mathcal{E}_{max})$ is quasi-abelian if and only if it is an AI-category.

[Brüstle, Hassoun, Shah, Tattar, Wegner]:

A pre-abelian category $\mathcal{A}$ is quasi-abelian if and only if it has admissible intersections.

Proof

$\Rightarrow$) Let $\mathcal{A}$ be a quasi-abelian additive category

$\Rightarrow$ $(\mathcal{A}, \underline{\mathcal{E}all})$ is an exact category

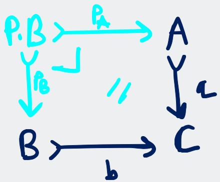
↳ exact structure by [Schneiders, 1999]

$\Rightarrow$ the class of ad monics are precisely the class of kernels and every ad monic is the kernel of its cokernel

We consider two ad monics (two ad subobjects) and we take their pull-back; it's the PB along kernels:

Proof

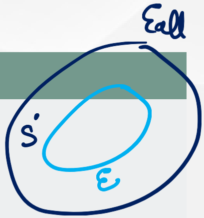
By [Kelly, 1969] p_A and p_B are also kernels.



So the P.B of ad monics is again ad and (AI) is satisfied ✓

Let $(\mathcal{A}, \mathcal{E})$ be an (AI)-exact category.
Suppose that $\mathcal{E} \subsetneq \mathcal{E}_{all}$

Proof

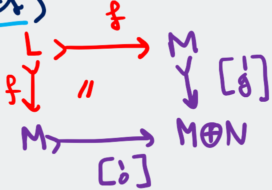


$S: \ 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$

We consider the P.B along the following two actions (ad monics)

$\mathcal{E}_{min} \subseteq \mathcal{E}$

So $(M, [\begin{smallmatrix}1\\g\end{smallmatrix}]) \cap_{\mathcal{E}} (M, [\begin{smallmatrix}1\\0\end{smallmatrix}]) =$



+ U.P

(L, f, f)

But f is not an ad monic!

$(f, g) \notin \mathcal{E} \Rightarrow$ ~~(A1)~~ $\Rightarrow \mathcal{E} = \mathcal{E}_{all}$

$\Rightarrow$ using (A2), $(A2)^{op}$ $\mathcal{A}$ is quasi-abelian by def.

By [Sch, 99']

∎

Abelian categories

[BHT, Theorem 4.22]:

An exact category $(\mathcal{A}, \mathcal{E})$ is an AIS-category if and only if $\mathcal{A}$ is abelian and $\mathcal{E} = \mathcal{E}_{all}$.

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[BHT, Theorem 3.7]

The following conditions are equivalent:

- $\mathcal{A}$ is an abelian category,
- $(\mathcal{A}, \mathcal{E}_{all})$ is an AIS-category,
- $Hom(\mathcal{A}) = Hom^{ad}(\mathcal{A})$,
- $Hom^{ad}(\mathcal{A})$ is closed under composition,
- $Hom^{ad}(\mathcal{A})$ is closed under addition.

Definition

We denote by $\mathcal{P}_X^{\mathcal{E}}$ the set of isomorphism classes of $\mathcal{E}$-subobjects of X. The relation

$$(Y, f) \leq (Z, g) \iff \exists \begin{array}{ccc} Y & \overset{\exists h}{\rightarrowtail} & Z \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle g} \\ & & X \end{array}$$

turns $(\mathcal{P}_X^{\mathcal{E}}, \leq)$ into a poset.

Generalised Intersection

Definition

Consider two $\mathcal{E}-$subobjects (A, f) and (B, g) of X. We denote the set of all common admissible subobjects of A and B as

$$Sub_X(A, B) := \{ (Y, h) \in P_X^{\mathcal{E}} \mid Y \in P_A^{\mathcal{E}},\ Y \in P_B^{\mathcal{E}} \}.$$

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[BHT, Definition 5.5]

We define the $\mathcal{E}-$intersection of (A, f) and (B, g) in $P_X^{\mathcal{E}}$ as

$$I_X(A, B) := Max(Sub_X(A, B)).$$

Generalised Sum

Definition

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[BHT, Definition 5.5]

We define the $\mathcal{E}-$sum of A and B in $P_X^{\mathcal{E}}$ as

$$Sum_X(A,B) := Min(Sup_X(A,B)).$$

Examples

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Consider the object $X = k^6 =< v_1, v_2, v_3, v_4, v_5, v_6 >$ and

$$V_1 =< v_1, v_2, v_3, v_4 > \quad \text{and} \quad V_2 =< v_2, v_3, v_4, v_5 > .$$

Examples

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$$V_1 =< v_1, v_2, v_3, v_4 > \quad \text{and} \quad V_2 =< v_2, v_3, v_4, v_5 > .$$

The abelian intersection $V_1 \cap V_2 =< v_2, v_3, v_4 >$ in $\mod k$

BUT

$$Int_X{}^{\mathcal{E}_{min}}(V_1, V_2) = Gr(2, 3)$$

and

$$Sum_X{}^{\mathcal{E}_{min}}(V1, V2) = \{(X, \mathbb{I})\}.$$

$\mathcal{E}-$Jacobson Radical

[BHT, Definition 6.1]

We define the $\mathcal{E}-$*Jacobson radical* to be the generalised intersection

$$rad_{\mathcal{E}}(X) := I_X\{(Y,f) \in \mathcal{S}_X \mid (Y,f) \in Max(\mathcal{S}_X)\}.$$

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[BHT, Proposition 6.2]

For all $X, Y \in$ and $R \xrightarrow{r} X$.

- For all $(R, r) \in rad(X), rad(Coker(r)) = \{0\}$.

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[BHT, Proposition 6.2]

For all $X, Y \in$ and $R \xrightarrow{r} X$.

- For all $(R, r) \in rad(X), rad(Coker(r)) = \{0\}$.
- For all $(Z, g) \in \mathcal{S}_X, Z$ is an $\mathcal{E}$-subobject of some $(R, r) \in rad(X)$ if and only if $pg = 0$ for all $\mathcal{E}-$simple quotients $p : X \twoheadrightarrow S$ of X.

Artin-Wedderburn exact structures

[BHT, Definition 6.4]:

An exact structure $\mathcal{E}$ on $\mathcal{A}$ is called *Artin-Wedderburn* if for any object $X \in \mathcal{A}$ the following properties are equivalent:

(AW1) Every sequence in $\mathcal{E}$ of the form $A \rightarrowtail X \twoheadrightarrow X/A$ splits,

(AW2) X is $\mathcal{E}-$semisimple,

(AW3) $rad(X) = \{0\}$.

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(AW2) X is $\mathcal{E}-$semisimple,

(AW3) $rad(X) = \{0\}$.

We say in this case that $(\mathcal{A}, \mathcal{E})$ is an $\mathcal{E}-$*Artin-Wedderburn category.*

Split Artin-Wedderburn structure

The split exact structure $\mathcal{E}_{min}$ is an *Artin-Weddernburn* exact structure:

[BHT, lemma 6.7]:

Any additive category $\mathcal{A}$ is an $\mathcal{E}_{min}$-Artin-Wedderburn category.

Krull-Schmidt categories

Definition

A category is *Idempotent complete* if every idempotent splits.

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A *Krull-Schmidt* category is a Hom-finite and Idempotent complete additive category.

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[BHT, Theorem 6.8]:

Let $(\mathcal{A}, \mathcal{E})$ be a Krull-Schmidt $\mathcal{E}$-Artin-Wedderburn category. Then $(\mathcal{A}, \mathcal{E})$ is a Jordan-Hölder exact category.

Nakayama Algebras

Definition

An *uniserial* module M is a module over a ring, such that ${\mathcal{P}_M}^{\mathcal{E}_{max}}$ is a totally ordered set.

Definition

A finite-dimensional algebra Λ is called *Nakayama* if every indecomposable right and left projective Λ-module is uniserial.

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[BHT, Theorem 6.13]:

Let Λ be a Nakayama algebra, and denote $\mathcal{A} = mod\Lambda$. Then an exact category $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}-$Artin-Wedderburn precisely when it is Jordan-Hölder.

$\mathcal{E}_{JH}-$length function

[BHT, Definition 7.1]

We define the $\mathcal{E}-$Jordan-Hölder length $l_{\mathcal{E}_{JH}}(X)$ of an object X in as the length of an $\mathcal{E}-$composition series of X. That is $l_{\mathcal{E}_{JH}}(X) = n$ if and only if there exists an $\mathcal{E}-$composition series

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \qquad \rightarrowtail X_{n-1} \rightarrowtail X_n = X$$

We say in this case that X is $\mathcal{E}-$finite.

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[BHT, Definition 7.1]

We define the $\mathcal{E}-$Jordan-Hölder length $l_{\mathcal{E}_{JH}}(X)$ of an object X in as the length of an $\mathcal{E}-$composition series of X. That is $l_{\mathcal{E}_{JH}}(X) = n$ if and only if there exists an $\mathcal{E}-$composition series

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail \dots \rightarrowtail X_{n-1} \rightarrowtail X_n = X$$

We say in this case that X is $\mathcal{E}-$finite.

Clearly, isomorphic objects have the same length, and therefore this definition gives rise to a length function $l_{\mathcal{E}} : Obj\mathcal{A} \to \mathbb{N} \cup \{\infty\}$ defined on isomorphism classes.

Properties of $l_{\mathcal{E}_{JH}}$

[BHT, Corollary 7.2]:

Let

$$X \rightarrowtail Z \twoheadrightarrow Y$$

be an admissible short exact sequence of finite length objects. Then

$$l_{\mathcal{E}}(Z) = l_{\mathcal{E}}(X) + l_{\mathcal{E}}(Y).$$

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[BHT, Proposition 7.8]:

If $\mathcal{E}$ and $\mathcal{E}'$ are exact structures on $\mathcal{A}$ such that $\mathcal{E}' \subseteq \mathcal{E}$, then $l_{\mathcal{E}'}(X) \leq l_{\mathcal{E}}(X)$ for all objects X in $\mathcal{A}$.

[BHT, Proposition 7.7 ($\mathcal{E}-$Hopkins–Levitzki theorem)]:

An object X of $(\mathcal{A}, \mathcal{E})$ is $\mathcal{E}-$Artinian and $\mathcal{E}-$Noetherian if and only if it has an $\mathcal{E}-$finite length.


THANK
YOU!