Pairwise Completability for 2-Simple Minded Collections

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based on joint works with Emily Barnard and Kiyoshi Igusa

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Outline

Set up

- Let Λ be a finite dimensional, basic algebra over an arbitrary field K.
- Denote by modΛ the category of finitely generate (right) modules
- All subcategories are assumed full and closed under isomorphisms.
- $(-)[1]$ is the shift functor.
- $S \in$ modΛ or $\mathcal{D}^b$(modΛ) is called a *brick* if $\mathrm{End}(S)$ is a division algebra. A collection of Hom-orthogonal bricks is a *semibrick*.

Key Definitions

Definition-Theorem (Brüstle-Yang '13)

Let $\mathcal{D}$ and $\mathcal{U}$ be semibricks in modΛ. Then $\mathcal{D} \sqcup \mathcal{U}[1]$ is called a *2-term simple minded collection* (2-SMC) if

1. $\mathrm{Hom}(\mathcal{D}, \mathcal{U}) = 0 = \mathrm{Ext}(\mathcal{D}, \mathcal{U})$
2. The closure of $\mathcal{D} \sqcup \mathcal{U}[1]$ under triangles and direct summands is all of $\mathcal{D}^b(\mathrm{mod}\Lambda)$.

- We say $\mathcal{D} \sqcup \mathcal{U}[1]$ is a *semibrick pair* if condition (1) holds.
- We say the semibrick pair $\mathcal{D} \sqcup \mathcal{U}[1]$ is *completable* if there exists a 2-SMC $\mathcal{D}' \sqcup \mathcal{U}'[1]$ so that $\mathcal{D} \subseteq \mathcal{D}'$ and $\mathcal{U} \subseteq \mathcal{U}'$.
- Natural question: under what conditions is a semibrick pair completable?

Some Examples

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

$$\Lambda = KQ/(\alpha\beta)$$

- $S_1 \sqcup S_2 \sqcup S_3$ and $S_1[1] \sqcup S_2[1] \sqcup S_3[1]$ are 2-SMCs.
- $P_1 \sqcup S_3 \sqcup S_2[1]$ is a 2-SMC.
- $P_1 \sqcup P_2[1]$ is a semibrick pair which is not completable.

Relationship to Lattices of Torsion Classes

Definition

Let $\mathcal{T}, \mathcal{F}$ be (full, closed under isomorphism) subcategories of modΛ.

1. Then $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$.
2. If $(\mathcal{T}, \mathcal{F})$ is a torsion pair, then $\mathcal{T}$ is called a *torsion class* and $\mathcal{F}$ is called a *torsion-free class*.

- If $\mathcal{T}$ (resp. $\mathcal{F}$) is a (full) subcategory which is closed under isomorphisms, extensions, and quotients (resp. submodules), then $(\mathcal{T}, \mathcal{T}^{\perp})$ (resp. $({}^{\perp}\mathcal{F}, \mathcal{F})$) is a torsion pair.
- The torsion classes (resp. torsion free classes) of modΛ form a lattice under containment [IRTT15].
- A *minimal inclusion* of torsion classes (resp. torsion free classes is a pair $\mathcal{T} \subseteq \mathcal{T}'$ so that $\mathcal{T} \subsetneq \mathcal{T}'' \subseteq \mathcal{T}'$ if and only if $\mathcal{T}'' = \mathcal{T}'$.

Relationship to Lattices of Torsion Classes

Definition-Theorem (Barnard-Carroll-Zhu '19)

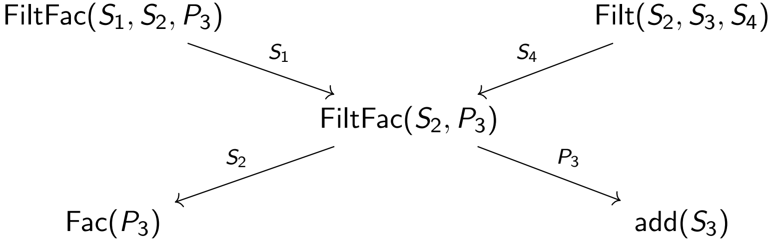
Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair.

1. A brick $S \in \text{mod}\Lambda$ is called a *minimal extending module* for $\mathcal{T}$ if there is a minimal inclusion $\mathcal{T} \subsetneq \text{Filt}(\mathcal{T} \cup \{S\})$.
2. A brick $S \in \text{mod}\Lambda$ is called a *minimal co-extending module* for $\mathcal{F}$ if there exists a minimal inclusion $\mathcal{F} \subsetneq \text{Filt}(\mathcal{F} \cup \{S\})$. Equivalently, S is a minimal extending module for ${}^{\perp}\text{Filt}(\mathcal{F} \cup \{S\})$.

An Example

$$Q = 1 \to 2 \to 3 \to 4$$

$$\Lambda = KQ$$



Observation: $S_2 \sqcup P_3 \sqcup S_1[1] \sqcup S_4[1]$ is a 2-SMC!

Relationship to Lattices of Torsion Classes

Theorem (Barnard-Carroll-Zhu '19)

Suppose $\mathcal{D}$ is a collection of bricks. Then $\mathcal{D}$ is the set of minimal extending modules (resp. minimal co-extending modules) for a torsion class (resp. torsion-free class) if and only if $\mathcal{D}$ is a semibrick.

Relationship to Lattices of Torsion Classes

From now on, we'll assume modΛ contains only finitely many torsion classes (i.e., Λ is τ-tilting finite [DIJ17]).

Theorem (Asai '19)

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. Let $\mathcal{D}$ be the set of minimal co-extending modules for $\mathcal{F}$ and let $\mathcal{U}$ be the set of minimal extending modules for $\mathcal{T}$. Then

1. $\mathcal{D} \sqcup \mathcal{U}[1]$ *is a 2-SMC.*
2. $\mathcal{T} = \mathrm{FiltFac}(\mathcal{D})$ *and* $\mathcal{F} = \mathrm{FiltSub}(\mathcal{U})$.
3. *The association* $(\mathcal{T}, \mathcal{F}) \mapsto \mathcal{D} \sqcup \mathcal{U}[1]$ *is a bijection between torsion pairs and 2-SMCs.*

Obstruction to Completability

Suppose $\mathcal{D} \sqcup \mathcal{U}[1]$ is the 2-SMC corresponding to a torsion pair $(\mathcal{T}, \mathcal{F})$.

- Let $S \in \mathcal{D}$. By the previous results, $\mathrm{Filt}(\mathcal{F} \cup \{S\})$ is a torsion-free class. Thus every proper submodule of S is in $\mathcal{F}$.
- Let $T \in \mathcal{U}$. By the previous results, $\mathrm{Filt}(\mathcal{T} \cup \{T\})$ is a torsion class. Thus every proper quotient of T is in $\mathcal{T}$.
- This implies that every nonzero morphism $T \to S$ must be mono or epi[1]. Otherwise the image would have no canonical exact sequence.

[1] This is also proven in [BY13].

Obstruction to Completability

Our non-example from before:

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

$$\Lambda = KQ/(\alpha\beta)$$

We said $P_1 \sqcup P_2[1]$ is a semibrick pair which is not completable. Notice the map $P_2 \to P_1$ has image S_2!

Question: Is this the only obstruction?

Answer: Sometimes! (e.g. hereditary algebras [IT] and Nakayama algebras [HIb]), but not always.

Where this Leaves Us

So far, we know that if $\mathcal{D} \sqcup \mathcal{U}[1]$ is a completable semibrick pair, then:

1. $\mathrm{Hom}(\mathcal{D},\mathcal{U}) = 0 = \mathrm{Ext}(\mathcal{D},\mathcal{U})$.
2. If $S \in \mathcal{D}$ and $T \in \mathcal{U}$, then every nonzero map $T \to S$ is mono or epi.

These are both *pairwise conditions*.

Definition

1. We say a semibrick pair $\mathcal{D} \sqcup \mathcal{U}[1]$ is *pairwise completable* if for all $S \in \mathcal{D}$ and $T \in \mathcal{U}$ there exists a 2-SMC $\mathcal{D}' \sqcup \mathcal{U}'[1]$ with $S \in \mathcal{D}'$ and $T \in \mathcal{U}'$.
2. We say Λ has the *pairwise completability property* if every pairwise completable semibrick pair is completable.

Known examples of algebras with this property are rep. finite hereditary algebras [IT] and Nakayama algebras [HIb, GM19].

Motivation

- Motivation for studying this pairwise property comes from the study of *picture groups* and *picture spaces*.
- The picture group of an algebra was first defined by Igusa-Todorov-Weyman [ITW] in the (representation finite) hereditary case and later generalized to τ-tilting finite algebras in [HIb].
- It is a finitely presented group whose relations encode the structure of the lattice of torsion classes.
- The corresponding picture space is the classifying space of the *(τ)-cluster morphism category* [IT, BM19] of the algebra.
- We show in [HIb] that the picture group and picture space have isomorphic (co-)homology when Λ has the pairwise completability property (plus one technical condition).

Mutation

Definition (Koenig-Yang '14)

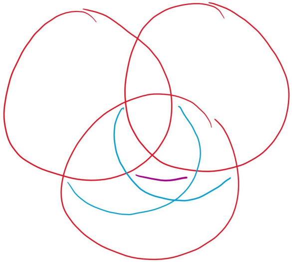
Let $\mathcal{D} \sqcup \mathcal{U}[1]$ be a 2-SMC and let $S \in \mathcal{D}$. Then there is a semibrick pair $\mu_S^+(\mathcal{D},\mathcal{U}[1])$, called the *left mutation* of $\mathcal{D} \sqcup \mathcal{U}[1]$ at S given as follows:

- Replace S with $S[1]$.
- For all $S' \in \mathcal{D}$ with $S' \neq S$, let $g^+_{SS'} : S'[-1] \to E_S$ be a minimal left (Filt$S$) approximation. Replace S' with cone$(g^+_{SS'})$. Note there is an exact sequence $S \hookrightarrow \mu_S^+(S') \twoheadrightarrow S'$.
- For all $T \in \mathcal{U}$, let $g^+_{ST} : T \to E_S$ be a minimal left (Filt$S$) approximation. If g^+_{ST} is mono, replace $T[1]$ with $\mathrm{coker}(g^+_{ST})$. If g^+_{ST} is epi, replace $T[1]$ with $\ker(g^+_{ST})[1]$.

Key observation: These are "pairwise formulas".

Mutation

1. From the perspective of torsion pairs, left mutation corresponds to traveling "down" the torsion lattice.
2. In the wall-and-chamber structure, this corresponds to crossing a wall. The 2-SMCs correspond to "*c*-vectors," which are normal to the walls bounding a chamber.



Completability and Mutation

Proposition (H.-Igusa '20)

Let $\mathcal{D} \sqcup \mathcal{U}[1]$ be a semibrick pair and let $S \in \mathcal{D}$.

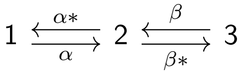
1. *If for all $T \in \mathcal{U}$ the minimal left $(\mathsf{Filt}S)$-approximation $T \to E_S$ is mono or epi, then $\mu_S^+(\mathcal{D} \sqcup \mathcal{U}[1])$ is a well-defined semibrick pair.*
2. *In this case, $\mathcal{D} \sqcup \mathcal{U}[1]$ is (pairwise) completable if and only if $\mu_S^+(\mathcal{D} \sqcup \mathcal{U}[1])$ is (pairwise) completable.*

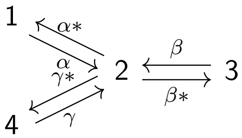
Since Λ is τ-tilting finite, if we continue to perform left mutations on semibrick pair, one of two things will happen.

1. We will reach a semibrick pair which fails to satisfy the mono/epi condition. In this case, our original semibrick pair is not completable.
2. We will reach a semibrick pair with $\mathcal{D} = \emptyset$. In this case, our original semibrick pair is completable.

Preprojective algebras

- Consider a Dynkin diagram W of type A, D, or E.
- Let Q be the quiver obtained by replacing each edge of W with a 2-cycle.
- The *preprojective algebra* of type W is the algebra $\Pi_W := KQ/I$, where I is generated by the sums of all 2-cycles sharing a source/target.





$A_3 : \alpha\alpha^*, \beta\beta^*, \alpha^*\alpha + \beta^*\beta$ $\qquad$ $D_4 : \alpha\alpha^*, \beta\beta^*, \gamma\gamma^*, \alpha^*\alpha + \beta^*\beta + \gamma^*\gamma$

Preprojective algebras

Theorem (Barnard-H.)

Let W be a Dynkin diagram of type A, D, or E. Then Π_W has the pairwise completability property if and only if $W = A_n$ with $n \leq 3$.

Idea of the proof:

1. Show directly that Π_W has the property if $W = A_n$ with $n \leq 3$ (or reference our later result!)
2. Reduce to the cases $W = A_4$ and $W = D_4$.
3. Substitute the algebra RA_4 (which has *all* 2-cycles as relations) for Π_{A_4}. This is a gentle algebra and has the same torsion lattice as Π_{A_4}[BCZ19, Miz14].
4. Use the relationship between completability and mutation to find counterexamples for RA_4 and Π_{D_4}.

Counterexample for RA_4

$$1 \underset{\longrightarrow}{\longleftarrow} 2 \underset{\longrightarrow}{\longleftarrow} 3 \underset{\longrightarrow}{\longleftarrow} 4$$

The semibrick pair $\mathcal{X} = (234) \sqcup (4)[1] \sqcup (320)[1]$ is pairwise completable, but not completable[2].

- Every 2-SMC contains $\mathrm{rk}(\Lambda)$ bricks [KY14], so $\mathcal{X}$ is not a 2-SMC.
- Mutating at (234) yields $(23) \sqcup (234)[1] \sqcup (320)[1]$ and the map $(320) \to (23)$ is neither mono nor epi. This means $\mathcal{X}$ is not completable. Similar arguments show that $\mathcal{X}$ is pairwise completable.
- Observation: The closure of the bricks in $\mathcal{X}$ under triangles is not all of $\mathcal{D}^b(\mathrm{mod}\Lambda)$, but their closure under extensions, kernels, and cokernels *is* all of $\mathrm{mod}\Lambda$...

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[2] 2 is the top and 4 is the socle of (234)

Why this is so strange

Recall that there are mutation-preserving bijections between the set of 2-SMCs and the following classes of objects:

- τ-tilting pairs
- sets of minimal extending modules
- 2-term silting complexes

All three of these classes are characterized by pairwise conditions!

Other known cases

Theorem

1. [Igusa-Todorov '17$^+$] *(Rep. finite) hereditary algebras have the pairwise completability property.*
2. [H.-Igusa '18$^+$] *Nakayama algebras have the pairwise completability property.*
3. [H.-Igusa '20] *A (τ-tilting finite) gentle algebra whose quiver contains no loops or 2-cycles has the pairwise completability property if and only if its quiver contains no vertex of degree 3 or 4.*

(Lack of) Patterns Amongst Examples/Nonexamples

There are both examples and nonexamples of the pairwise completability property in the following classes of algebras:

- Representation finite algebras
- (τ-tilting finite) tame algebras
- (τ-tilting finite) cluster-tilted algebras

Moreover, the property is not stable under quotients.

A Rank 3 Pattern Emerges

The counterexamples appearing in our studies of gentle algebras and preprojective algebras have been semibrick pairs $\mathcal{D} \sqcup \mathcal{U}[1]$ satisfying $|\mathcal{D}| + |\mathcal{U}| = 3 < \mathrm{rk}(\Lambda)$. It turns out this is not an accident:

Theorem (Barnard-H.)

Let Λ be a τ-tilting finite algebra with $\mathrm{rk}(\Lambda) \leq 3$. Then Λ has the pairwise 2-simple minded completability property.

Theorem (Barnard-H.)

Let Λ be any τ-tilting finite algebra. Then the following are equivalent.

1. *Λ has the pairwise 2-simple minded completability property.*
2. *Every pairwise completable semibrick pair $\mathcal{D} \sqcup \mathcal{U}[1]$ which satisfies $|\mathcal{D}| + |\mathcal{U}| = 3$ is completable.*

"Full-size" semibrick pairs

The key to the rank 3 case was that if $\mathrm{rk}(\Lambda) = 3$, then any semibrick pair of size 3 is a 2-SMC.

Conjecture

Let Λ be a τ-tilting finite algebra of rank n. Then any semibrick pair $\mathcal{D} \sqcup \mathcal{U}[1]$ with $|\mathcal{D}| + |\mathcal{U}| = n$ is a 2-SMC.

This conjecture would imply that $\mathrm{rk}(\Lambda)$ is an upper bound on the size of a semibrick pair (when Λ is τ-tilting finite).

This is (very) false in the τ-tilting infinite case:

- Over a tame hereditary algebra, any finite collection of homogeneous bricks is a semibrick.
- Tame hereditary algebras can even have pairwise completable semibrick pairs of size $\mathrm{rk}(\Lambda)$ which are not completable.

Preliminary result

Theorem (Barnard-H.)

Let $n \in \mathbb{N}$ and let $\mathcal{D} \sqcup \mathcal{U}[1]$ be a semibrick pair for Π_{A_n} with $|\mathcal{D}| + |\mathcal{U}| = n$. Then $\mathcal{D} \sqcup \mathcal{U}[1]$ is a 2-SMC.

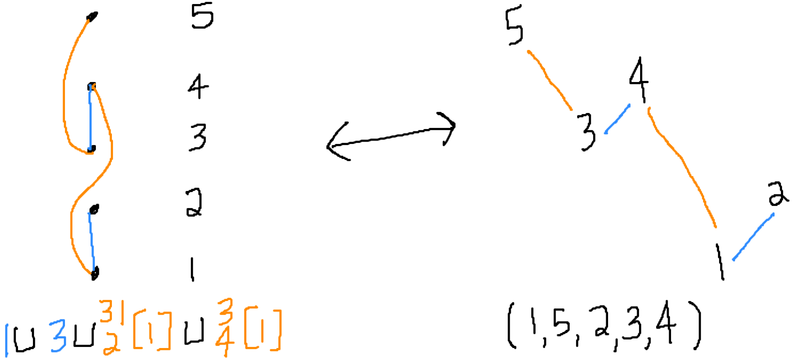
Idea of the proof:

1. As before, we work over RA_n instead of Π_{A_n}.
2. The torsion lattice is isomorphic to the weak order on the Coxeter group A_n (the group of permutations on $n+1$ letters) [BCZ19].
3. The *canonical join representations* (the bricks in $\mathcal{D}$) and the *canonical meet representations* (the bricks in $\mathcal{U}$) are separately encoded by *arc diagrams* [Rea15, BCZ19].

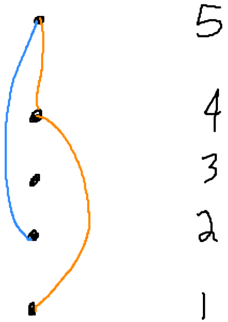
(continued on next slide)

Preliminary result

4. We define *2-colored arc diagrams* to encode both sets of bricks simultaneously and show a collection of n arcs always defines a permutation in A_n (and hence a 2-SMC).



2-colored arc diagram for the counterexample



$$\begin{matrix}2\\3\\4\end{matrix} \sqcup 4[1] \sqcup \begin{matrix}3\\2\\1\end{matrix}[1]$$

Reformulation in terms of wide subcategories

Recall that a (full, closed under isomorphism) subcategory $W \subseteq \text{mod}\Lambda$ is called *wide* if it is closed under extensions, kernels, and cokernels.

Assume Λ is τ-tilting finite.

- Ringel [Rin76] showed that every wide subcategory of $\text{mod}\Lambda$ is of the form $\text{Filt}(\mathcal{D})$ for $\mathcal{D}$ a semibrick.
- A result of Jasso [Jas14] (see also [DIR$^+$]) further shows that $\text{Filt}(\mathcal{D})$ is equivalent to $\text{mod}\Lambda'$ for some τ-tilting finite algebra Λ' of rank $|\mathcal{D}|$.
- This means wide subcategories have their own 2-SMCs and semibrick pairs!

Reformulation in terms of wide subcategories

Let $W \subseteq \text{mod}\Lambda$ be a wide subcategory.

- Any semibrick in W is also a semibrick in $\text{mod}\Lambda$.
- Any semibrick pair for W is also a semibrick pair for $\text{mod}\Lambda$.
- Natural question: If $\mathcal{D} \sqcup \mathcal{U}[1]$ is a 2-SMC for W, is it completable as a semibrick pair for $\text{mod}\Lambda$?

Reformulation in terms of wide subcategories

- Observation: Suppose $\mathcal{D} \sqcup \mathcal{U}[1]$ and $\mathcal{D}' \sqcup \mathcal{U}'[1]$ are semibrick pairs (for modΛ) related by a sequence of mutations. Then any wide subcategory containing the bricks in $\mathcal{D} \sqcup \mathcal{U}$ also contains the bricks in $\mathcal{D}' \sqcup \mathcal{U}'$
- Consequence: Suppose $\mathcal{D} \sqcup \mathcal{U}[1]$ is completable (as a semibrick pair for modΛ) and let W be the smallest wide subcategory which contains the bricks in $\mathcal{D} \sqcup \mathcal{U}$. Then $\mathcal{D} \sqcup \mathcal{U}[1]$ is a 2-SMC for W.

Reformulation in terms of wide subcategories

Putting this together, our conjecture about full-rank semibrick pairs would imply the following:

Conjecture

1. *Let $\mathcal{D} \sqcup \mathcal{U}[1]$ be a semibrick pair for* modΛ *and let W be the smallest wide subcategory containing the bricks in $\mathcal{D} \sqcup \mathcal{U}$. Then the following are equivalent.*
   1. *$\mathcal{D} \sqcup \mathcal{U}[1]$ is completable.*
   2. *$\mathcal{D} \sqcup \mathcal{U}[1]$ is a 2-SMC for W.*
   3. $\mathrm{rk}(W) = |\mathcal{D}| + |\mathcal{U}|$.
2. *The following are equivalent for all τ-tilting finite algebras Λ.*
   1. *Λ has the pairwise completability property.*
   2. *If $\mathcal{D} \sqcup \mathcal{U}[1]$ is a semibrick pair (for* modΛ*) of rank 3 and for $S \in \mathcal{D}$ and $T \in \mathcal{U}$ the smallest wide subcategory containing S and T has rank 2, then the smallest wide subcategory containing the bricks in $\mathcal{D} \sqcup \mathcal{U}$ has rank 3.*

Thank you!!

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