Descending chains of coprime pairs and the exchange property

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I.a. Right cotorsion rings

Def.- A module M_R is cotorsion if $\mathrm{Ext}^1_R(F,M)=0$ $\forall F_R$ flat.

Cotorsion modules where introduced by Harrison in 1959 as a homological generalization of pure-injective rings.

For instance, for abelian groups, they are just the (not necessarily pure) quotients of pure-inj. groups

Def.- A ring R is right cotorsion if so is R_R

Facts

1. R_R is cotorsion iff it is the endomorphism ring of a flat cotorsion module.

2. Flat cotorsion modules in Mod-R are just the pure-injective objects in Flat-R (but not necessarily in Mod-R)

Def.- Let $\mathcal{A}$ be an additive category with direct limits. An object $A \in \mathcal{A}$ is called finitely presented if $\mathrm{Hom}(A,-)$ commutes with direct limits.

Def.- An additive category $\mathcal{A}$ with direct limits is called *finitely accessible* if there exists a set $\mathcal{A}_0$ of finitely presented objects of $\mathcal{A}$ such that $\mathcal{A} = \varinjlim \mathcal{A}_0$.

(Finitely) accessible categories have been studied, for instance, by Adámek, Crawley-Boevey, Makkai, Paré, Prest, Rosicky, ...

Prop.- Any additive finitely accessible category is equivalent to Flat-R for some ring with enough idempotents R.

Moreover the solution to the flat cover conjecture gives as a byproduct:

Theorem (El Bashir, Enochs, Biçan)

For every right module M, there exists a monomorphism $u : M \longrightarrow C$ into a cotorsion module s.t.

i) Any other morphism $f : M \longrightarrow C'$, with C' cotorsion factors through u.

ii) u is minimal, in the sense that if $f : C \to C$ satisfies that $f \circ u = u$, then f is an automorphism

Corollary.- Define that

$$0 \to A \to B \to C \to 0$$

is pure-exact in an additive finitely accessible $\mathcal{A}$ category when is a direct limit of splitting sequences. Then we get:

Corollary Any object has a pure-injective envelope in $\mathcal{A}$. Moreover, its endomorphism ring is a right cotorsion ring.

Problem. Study the structure of these rings.

Difficulty Usual techniques in purity, such as Functor Categories, finite definition subgroups or Model Theory seem to fail in this setup.

Example. Let R be a right perfect ring which is not right pure-injective.

R_R is trivially (Σ-)cotorsion, since every flat right module is projective.

Assume it would be possible to construct a full embedding

$$U : \text{Flat-}R \longrightarrow \mathcal{B}$$

in a Grothendieck category $\mathcal{B}$ s.t. R become injective in $\mathcal{B}$. Then:

$$R_R = End_R(R) = End_{\mathcal{B}}(U(R))$$

and R_D would be right p.i., since endomorphism rings of injective objects in Grothendieck categories are pure-injective. But there exist non right p.i right cotorsion rings.

Idea (G. Herzog) Endomorphism rings of pure-injective module are (von Neumann) regular and right self-inj. modulo their Jacobson radical. And regular rings are plenty of idempotents.

Develope a method to construct idempotents in right cotorsion rings.

Theorem (G., Gómez Pardo). Let M be an indecomposable module and assume any directed union of direct summands of $M^{(I)}$ is a direct summand for any set I. Then $\mathrm{End}(M)$ is local.

Applications:

- Decompositions of modules into direct summands
- Recently used by Positselski and Stovicek to study contramodules associated to cosheaves.

In our setting: Choose an element $a \in R$.

- Clearly $Ra + R(1-a) = R$. So if $Ra \cap R(1-a) = 0$, then ${}_R R = Ra \oplus R(1-a)$ and we got our idempotent

- Otherwise, $Ra^2 \subseteq Ra$ and $R(1-a^2) \subseteq R(1-a)$ and again $R = Ra^2 + R(1-a^2)$. We may hope that

$$Ra^2 \cap R(1-a^2) < Ra \cap R(1-a).$$

Continuing in this fashion, we get commutative diagrams with splitting rows,

$$
\begin{tikzcd}
R_R \arrow[r, "{(a,\,1-a)}"] \arrow[d, equal] & R_R \oplus R_R \arrow[d, "{\begin{pmatrix} a & 0 \\ 0 & 1+a \end{pmatrix}}"] \\
R_R \arrow[r, "{(a^2,\,1-a^2)}"] \arrow[d, equal] & R_R \oplus R_R \arrow[d, "{\begin{pmatrix} a^2 & 0 \\ 0 & 1+a^2 \end{pmatrix}}"] \\
R_R \arrow[r, "{(a^4,\,1-a^4)}"] & R_R \oplus R_R
\end{tikzcd}
$$

whose direct limit would be of the form

$$\Sigma = \; R_R \xrightarrow{u} F_1 \oplus F_2 \xrightarrow{p} G \longrightarrow 0$$

with F_1, F_2, G flat. So it splits since R_R is cotorsion

Call $\pi : F_1 \oplus F_2 \longrightarrow R$ the splitting. Then,
if $(x, y) = u(1)$, call $a_w = \pi(x)$, $b_w = \pi(y)$.
Then $Ra_w + Rb_w = R$. And thus, we have
a splitting sequence

$$R_R \xrightarrow{(a_w, b_w)} R_R \oplus R_R$$

below the diagram.

Continue transfinitely in this fashion and show that the process stops by cardinality. And show that when the process ~~stops~~ stabilizes, say in (a_α, b_α), then ${}_R R = Ra_\alpha \oplus Rb_\alpha$

We may formalize this idea as follows:

__Def.__ A pair of elements $a, b \in R$ are called a left coprime pair if $Ra + Rb = R$. Denote it by $\langle a, b \rangle$

Our prototype is $\langle a, 1-a \rangle$

We will say that $\langle a,b\rangle \leq \langle a',b'\rangle$ if $Ra \subseteq Ra'$ and $Rb \subseteq Rb'$.

Theorem (G., Herzog). Let R be a right cotorsion ring. And let $\langle a,b\rangle$ be a left coprime pair. Then there exists an idempotent $e \in R$ s.t. $\langle e, 1-e\rangle \leq \langle a,b\rangle$

Moreover, $\langle e, 1-e\rangle$ is minimal in the former order relation.

Following these idea, we were able to prove:

Theorem Let R be a right cotorsion ring. Then R/J is von Neumann regular and right self-injective and idempotents lift modulo J.

Indeed our results suggest a possible extension of Ziegler spectrum to additive finitely accessible categories:

Def. - Let $\{X_i\}_I$ be a family of modules and $u : M \rightarrowtail \prod_I X_i$, an embedding. u is strongly pure if for any ${}_R L$, the canonical morphism

$$M \otimes_R L \longrightarrow \left(\prod_I X_i\right) \otimes L \longrightarrow \prod_I (X_i \otimes L)$$

is a monomorphism.

Theorem (G., Rothmaler) A submodule M of a product of flat modules is flat iff it is strongly pure.

Theorem (G., Herzog). There exist a set $\{U_i\}_I$ of indecomposable pure-injective objects in Flat-R s.t. a right R-module M lives in Flat-R iff it is a strongly pure submodule of a product of them.

Remark The proof of the theorem is inspired by Auslander's objects defined by functors.

II.6. Exchange rings

Def.- ((Crawley and Jonsson '64, Warfield Jr. '72)

A right R module M satisfies the (finite) exchange property if for any module X and decompositions

$$X = M' \oplus Y = \bigoplus_{i \in I} N_i$$

with $M' \cong M$ (resp., I finite), there exist submodules $N_i' \leq N_i$ s.t. $X = M' \oplus \left(\bigoplus_I N_i'\right)$.

II. Exchange rings

Theorem.- (Nicholson '77) R_R satisfies the (finite) exchange property iff so does ${}_RR$.

Theorem (Nicholson '77, Goodearl '76) R_R satisfies the finite exchange property iff $\forall a \in R\ \exists e = e^2 \in R$ s.t. $eR \subseteq aR$ and $(1-e)R \subseteq (1-a)R$

Def.- These rings are called exchange rings.

Thus, our construction was implicitely giving a method to prove that a ring is exchange

However, there are many exchange rings which are not ω-torsion. So it would be interesting to study:

I) For which exchange rings the above construction can be done.

II) What additional properties we get.

Example of exchange rings

I) Right self-injective rings

II) Right pure-injective rings

III) Right continuous rings

IV) Local rings

V) Right perfect rings.
In particular, right artinian rings.

VI) (von Neumann) regular rings

VII) Semiperfect rings

VIII) Semiregular rings

IX) Right cotorsion rings

X) Rings invariant under automorphisms of their injective envelope

Theorem (Warfield) M_R satisfies the finite exchange property iff $\mathrm{End}_R(M)$ is an exchange ring.

Main open question

Does the finite exchange property in M_R imply the general one?. True if M_R is finitely generated.

Open for almost 60 years.

III. Descending chains of coprime pairs

Def. - Let $\mathcal{C} = \{ < a_\alpha, b_\alpha > \}_{\alpha < \gamma}$ be a descending chain of right coprime pairs. We will say that $\mathcal{C}$ is compatible if there exist pairs of scalars $(r_{\alpha,\beta}, s_{\alpha\beta})$ for every $\alpha \leq \beta < \gamma$ s.t.

i) $a_\beta = r_{\alpha\beta} a_\alpha$ and $b_\beta = s_{\alpha\beta} b_\alpha$ if $\alpha \leq \beta$

ii) $r_{\alpha\delta} = r_{\beta\delta} r_{\alpha\beta}$ and $s_{\alpha\delta} = s_{\beta\delta} s_{\alpha\beta}$

if $\alpha \leq \beta \leq \delta < \gamma$

Def.- A ring R is right strongly exchange if for any compatible descending chain of right coprime pairs $\mathcal{C}$, there exist a minimal right coprime pair seated below it.

Examples

I. Any left self-injective ring is right strongly exchange.

II. Any left pure-injective ring is right strongly exchange.

In particular $\mathrm{End}_R(E)$ is right strongly exchange for any (pure-) injective or quasi-injective module M.

III. Any local ring is right strongly exchange.

III. Let V a vector space with $\dim V = \aleph_0$

Then $S = \mathrm{End}_K(V)$ is regular and right self-injective. Therefore, it is left strongly exchange. However, it is not right strongly exchange:

Let $B = \{v_1, \ldots, v_n, \ldots\}_{n\in\mathbb{N}}$ be a basis and call $e_i : V \longrightarrow V$ the endomorphisms

$e_i(v_j) = \delta_{ij} v_j \quad \forall j \in \mathbb{N}$

Let $p : V \longrightarrow V$ defined as $p(v_i) = v_1 \ \forall i \in \mathbb{N}$.

Then $\{(1 - \sum_{i=1}^{n} e_i, 1 - p)\}_{n \in \mathbb{N}}$ is a descending chain of right coprime pairs which does not have any minimal one below it.

In particular:

- strongly exchange is not a left-right symmetric condition
- Regular rings don't need to satisfy it.

IV. Any right cotorsion ring is left strongly exchange.

V. A module M_R is said continuous if:

i) Any $N \leq M_R$ essentially embeds in a direct summand of M.

ii) If $N \leq M_R$ is isomorphic to a direct summand of M_R, then $N \leq_{\oplus} M$.

The notion of continuous module has its origin in von Neumann continuous geometries and play an important rule in the structure of (von Neumann) regular rings.

Indeed, for regular rings, condition II is always satisfied

- Any right continuous ring is left strongly exchange.

VI. Left perfect rings are right strongly exchange.

Note.- All the above classes of rings satisfy that R/J is regular and idempotents lift modulo J.

This suggests that this might be a general property of strongly exchange rings.

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This would allow to get a unified study of them.

IV. Structure of right strongly exchange rings

The following result is clear

Prop. Any right strongly exchange ring is an exchange ring.

Theorem (G., Izurdiaga) Let R be a right strongly exchange ring. Then R/J is regular and idempotents lift modulo J.

Idea of the proof

- Characterize first J as the elements $a \in R$ s.t. the only minimal right coprime pair below $\langle a, 1-a \rangle$ is $\langle 0, 1 \rangle$.
- Now use the existence of minimal coprime pairs to show that R/J is regular by constructing idempotents. This is the hardest part.
- Finally, show that idempotents lift. $\square$

Remark If R_R is continuous, then R/J does not need to be right self-injective.

Therefore:

- Right strongly exchange rings do not need to be left self-injective modulo their Jacobson radical.
- However, if R is left continuous, so is R/J.

- We do not know whether a right strongly exchange ring R satisfies that R/J is left continuous.

- However, this is the case if we strengthen a little bit the definition. But then, we lose some example of right strongly exchange rings.

Theorem Assume that for every descending system of compatible right coprime pairs, there exists a minimal right coprime pair below it. Then R/J is left continuous.

Note. This is also the case for countable right strongly exchange rings. (even if we only assume that they have countable uniform dimension)

VI Open questions

I Let R_R be a right strongly exchange ring. Does it satisfy the above condition?

I.e., does there exist a minimal right coprime pair below any compatible descending system of right coprime pairs?

II. Let R be a right strongly exchange ring.

Is R/J left continuous?

III. Recently, Lam, Khurana and Nielsen have proved that if R is an exchange ring, then for any element $a \in R$, the idempotent below $\langle a, 1-a \rangle$ can be chosen with a "two-sided property", in the sense that:

$$e \in aR \cap Ra \quad \text{and} \quad 1-e \in (1-a)R \cap R(1-a)$$

The fact that being strongly exchange is not a left-right symmetric property shows that we cannot expect a similar result.

However, proofs frequently involve some left-right symmetric properties. It would be interesting clarify this point.

IV. Exchange rings are related to the study of the unimodular equation

$$ax + by = 1$$

For instance, for square matrices.

It would be interesting to study infinite systems of unimodular equations in terms of strongly exchange properties.