Interpretation Functors II

Lorna Gregory

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Università della Campania

Recap

Let R be a ring. A **pp-n-formula** (over R) is a formula

$$\varphi(\overline{x}) := \exists \overline{y} \ (\overline{x} \ \overline{y})A = 0$$

where A is an $(n+m) \times l$ matrix with entries in R, $\overline{x}$ an n-tuple of variables and $\overline{y}$ an m-tuple of variables. Write pp_R^n for the set of pp-n-formulae over R.

For M an R-module and $\varphi \in \mathrm{pp}_R^n$, we write $\varphi(M)$ for the solution set of φ in M.

A full subcategory $\mathcal{D} \subseteq \text{Mod-}R$ is called a **definable subcategory** if it is the form

$$\mathcal{D} = \{M \in \text{Mod-}R \mid \varphi_i(M) = \psi_i(M) \text{ for all } i \in I\}$$

where φ_i, ψ_i are pairs of pp-formulae indexed by I.

Interpretation functors

Definition

Let R, S be rings and $\mathcal{D}$ a definable subcategory of Mod-S.
Suppose that φ, ψ are pp-n-formulae over S and that for each $r \in R$, $\rho_r(\overline{x_1}, \overline{x_2})$ is a pp-$2n$-formula in variables $\overline{x_1}, \overline{x_2}$ each of length n.
Suppose that for all $M \in \mathcal{D}$ the following hold:

1. $\varphi(M) \supseteq \psi(M)$
2. for all $r \in R$, $\rho_r(\overline{x_1}, \overline{x_2})$ defines an endomorphism ρ_r^M of the abelian group $\varphi(M)/\psi(M)$
3. $\varphi(M)/\psi(M)$ equipped with the ρ_r^M actions is an R-module.

Then $(\varphi, \psi, (\rho_r)_{r \in R})$ defines an additive functor

$$I : \mathcal{D} \longrightarrow \text{Mod-}R, \quad M \mapsto (\varphi(M)/\psi(M), (\rho_r^M)_{r \in R}).$$

We call any such functor an **interpretation functor**.

Let R, S be rings.

Theorem (Krause, Prest)

Let $\mathcal{D}$ be a definable subcategory of Mod-S. *An additive functor $I : \mathcal{D} \longrightarrow$* Mod-$R$ *is an interpretation functor if and only if I commutes with direct limits and products.*

Interpretation functors - Examples coming from algebra.

Let R, S be rings. Let ${}_R B_S$ be an R-S-bimodule.

- If B_S is finitely presented then $\mathrm{Hom}_S(B, -) : \text{Mod-}S \to \text{Mod-}R$ is an interpretation functor.
- If ${}_R B$ is finitely presented then $- \otimes_R B : \text{Mod-}R \to \text{Mod-}S$ is an interpretation functor.
- If $B_S \in \mathbf{FP}_2$ then $\mathrm{Ext}_S(B, -) : \text{Mod-}S \to \text{Mod-}R$ is an interpretation functor.
- If ${}_R B \in \mathbf{FP}_2$ then $\mathrm{Tor}_R(B, -) : \text{Mod-}R \to \text{Mod-}S$ is an interpretation functor.

In particular, the equivalences coming from classical tilting are interpretation functors between definable subcategories.

Let R, S be rings.

Proposition

Let $I : \text{Mod-}S \to \text{Mod-}R$ *be an interpretation functor such that* $\langle I\text{Mod-}S \rangle = \text{Mod-}R$. *There is an* $n \in \mathbb{N}$ *and a lattice embedding* $i : \text{pp}_R^1 \hookrightarrow \text{pp}_S^n$.

Reminder: An R-module M is **pure-injective** if any system of (inhomogeneous) linear equations over R, in arbitrary many variables, which is finitely solvable in M, has a solution in M.

Remark

Let $I : \text{Mod-}S \to \text{Mod-}R$ *be an interpretation functor. If* $M \in \text{Mod-}S$ *is pure-injective then* IM *is pure-injective.*

Theorem (G.)

Let $I : \text{Mod-}S \to \text{Mod-}R$ *be an interpretation functor such that* I *maps finitely presented* S*-modules to finitely presented* R*-modules. If* I *is full on finitely presented* S*-modules then* I *is full on pure-injective* S*-modules.*

Finite-dimensional algebras

Conjecture (Prest 80's)

Let $\mathcal{A}$ be a finite-dimensional k-algebra. If $\mathcal{A}$ is of wild representation type then the theory of $\mathcal{A}$-modules interprets the theory of $k\langle x, y\rangle$-modules.

Hence, if k is countable, $\mathcal{A}$ has undecidable theory of modules.

Conversely, if $\mathcal{A}$ is tame then the theory of $\mathcal{A}$-modules is decidable.

What does “theory of $\mathcal{A}$-modules” mean?

A (first order) **sentence in the language of $\mathcal{A}$-modules** is a statement, which can be assigned a truth value, built up from homogenous linear equations over $\mathcal{A}$ in variables $\{x_i \mid i \in \mathbb{N}\}$, $\exists x_i$, $\forall x_i$, `NOT`, `AND` and `OR`.

Examples: Let $r, s \in \mathcal{A}$.

$$\texttt{NOT}(\forall x_1 \exists x_2 \exists x_3 \; x_1 + x_2 \cdot r + x_3 \cdot s = 0)$$

is a sentence in the language of $\mathcal{A}$-modules.

$$\forall x_1 \; \texttt{AND} \; x_2 \cdot r \qquad \text{and} \qquad x_1 + x_2 \cdot r = 0$$

are not sentences in the language of $\mathcal{A}$-modules.

The **theory of $\mathcal{A}$-modules** is the set of all sentences in the language of $\mathcal{A}$-modules which are true in all $\mathcal{A}$-modules.

↷

From now on k is an algebraically closed field.

Definition

A finite-dimensional k-algebra $\mathcal{A}$ is **wild** if:
there exists a **representation embedding**

$$F : \text{fin-}k\langle x, y\rangle \to \text{fin-}\mathcal{A}$$

i.e. F is an exact k-linear functor which reflects isomorphism classes and sends indecomposable modules to indecomposable modules.
Equivalently, for every finite-dimensional k-algebra $\mathcal{B}$ there exists a representation embedding

$$F : \text{fin-}\mathcal{B} \to \text{fin-}\mathcal{A}.$$

↷

Finite-dimensional algebras

Conjecture (Prest 80's)

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Hence, if k is countable, $\mathcal{A}$ has undecidable theory of modules.

Conversely, if $\mathcal{A}$ is tame then the theory of $\mathcal{A}$-modules is decidable.

Definition

A finite-dimensional k-algebra $\mathcal{A}$ is **tame** if, for every dimension $d \in \mathbb{N}$, there is a finite number of $\mathcal{A}$-$k[x]$-bimodules $M_1, \ldots, M_{u(d)}$, which are finitely generated and free as $k[x]$-modules, such that almost all d-dimensional indecomposable $\mathcal{A}$-modules are of the form

$$M_i \otimes_{k[x]} k[x]/\langle x - \lambda \rangle$$

for some $1 \leq i \leq u(d)$ and some $\lambda \in k$.

Theorem (Drozd)

Every finite-dimensional k-algebra is either tame or wild.

Definition

Let $\mu(d)$ be the least possible value of $u(d)$ in the definition of a tame algebra. The finite-dimensional k-algebra $\mathcal{A}$ is **tame domestic** if $\mu(d)$ is bounded.

Theorem (G., Prest)

Let $\mathcal{A}, \mathcal{B}$ be finite-dimensional k-algebras. If $I : \text{Mod-}\mathcal{A} \to \text{Mod-}\mathcal{B}$ is a k-linear interpretation functor and $\langle I\text{Mod-}\mathcal{A}\rangle = \text{Mod-}\mathcal{B}$ then:

- *if $\mathcal{A}$ is tame then $\mathcal{B}$ is tame*
- *if $\mathcal{A}$ is tame domestic then $\mathcal{B}$ is tame domestic*
- *if $\mathcal{A}$ is of polynomial growth then $\mathcal{B}$ is of polynomial growth*
- *if $\mathcal{A}$ is of non-exponential growth then $\mathcal{B}$ is of non-exponential growth*

Moreover, if $\mathcal{A}$ is wild then there exists a k-linear interpretation functor $I : \text{Mod-}\mathcal{A} \to \text{Mod-}\mathcal{B}$ such that $\langle I\text{Mod-}\mathcal{A}\rangle = \text{Mod-}\mathcal{B}$.

Corollary

A finite-dimensional k-algebra $\mathcal{A}$ is wild if and only if for every finite-dimensional k-algebra $\mathcal{B}$ there is a k-linear interpretation functor $I : \text{Mod-}\mathcal{A} \to \text{Mod-}\mathcal{B}$ such that $\langle I\text{Mod-}\mathcal{A}\rangle = \text{Mod-}\mathcal{B}$.

Wild implies undecidability

A finite dimensional k-algebra $\mathcal{A}$ is **finitely controlled wild** if there is a representation embedding

$$F : \text{fin-}k\langle x, y\rangle \longrightarrow \text{fin-}\mathcal{A}$$

and $C \in \text{fin-}k\langle x, y\rangle$ such that for all $N, M \in \text{fin-}k\langle x, y\rangle$

$$\text{Hom}_{\mathcal{B}}(FM, FN) = F\text{Hom}(M, N) \oplus \text{Hom}(FM, FN)_C$$

where $\text{Hom}(FM, FN)_C$ is the set of maps which factor through some C^n.

Theorem (G., Prest)

Let $\mathcal{A}$ be a finite-dimensional k-algebra. If $\mathcal{A}$ is finitely controlled wild then there is a k-linear essentially surjective interpretation functor $I : \text{Mod-}\mathcal{A} \to \text{Mod-}k\mathbb{K}_3$.

Corollary

In the above situation, the theory of $\mathcal{A}$-modules interprets the theory of $k\mathbb{K}_3$-modules. In particular, if k is countable, the theory of $\mathcal{A}$-Mod is undecidable.