Interpretation functors I

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Model theory of modules

A (first order) **formula in the language of R-modules** is an expression, built up from homogenous linear equations over R in variables $\{x_i \mid i \in \mathbb{N}\}$, $\exists x_i$, $\forall x_i$, NOT, AND and OR which makes sense when appropriate (=free) variables are replaced by elements of a module.

Example: Let $R := \mathbb{Z}$. Then

$$(1)\ \ (x_1 \cdot 2 = 0)\ \mathtt{OR}\ (\exists x_3\ x_3 \cdot 3 = x_2)\ \ \text{and}\quad (2)\ \forall x_1\ (x_1 \cdot 4 = 0)$$

are formulae.

A formula which has no free variables/can be assigned a truth value in every R-module is called a **sentence in the language of R-modules**.

The **theory of R-modules** is the set of all sentences in the language of R-modules which are true in all R-modules.

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Model theory of modules

A **pp-n-formula** (over R) is a formula $\varphi(\overline{x})$ of the form

$$\exists \overline{y} \bigwedge_{i=1}^{l} \sum_{j=1}^{n} x_j r_{ij} + \sum_{k=1}^{m} y_k s_{ik} = 0$$

where $r_{ij}, s_{ik} \in R$. Write pp_R^n for the set of pp-n-formulae over R.

For $M \in \text{Mod-}R$, we write $\varphi(M)$ for the solution set of φ in M.

After identifying $\varphi, \psi \in \text{pp}_R^n$ such that $\varphi(M) = \psi(M)$ for all $M \in \text{Mod-}R$, pp_R^n is a lattice under the order $\psi \leq \varphi$ if $\psi(M) \subseteq \varphi(M)$ for all R-modules M.

We write $\varphi + \psi$ for the join (l.u.b) in this lattice and $\varphi \wedge \psi$ for the meet (g.l.b).

For $M \in \text{Mod-}R$, $(\varphi + \psi)(M) = \varphi(M) + \psi(M)$ and $(\varphi \wedge \psi)(M) = \varphi(M) \cap \psi(M)$.

Model theory of modules

Let $\psi \leq \varphi \in \mathrm{pp}_R^1$ and $b \in \mathbb{N}$. We write

$$|\varphi/\psi| > b$$

for the sentence in the language of R-modules expressing in all modules M that

$$|\varphi(M)/\psi(M)| > b.$$

The Baur-Monk Theorem

Every formula in the language of R-modules is equivalent to a boolean combination of pp-formulae and sentences

$$|\varphi/\psi| > b$$

where $b \in \mathbb{N}$ and φ, ψ are pp-1-formulae such that $\psi \leq \varphi$.

Model theory of modules

Definition

An R-module M is **pure-injective** if any system of (inhomogeneous) linear equations over R, in arbitrary many variables, which is finitely solvable in M, has a solution in M.

Theorem

- *For every R-module M there exists a pure-injective R-module M' which is* **elementary equivalent** *to M, that is, for every sentence χ, χ is true in M if and only if χ is true in M'.*
- *For every R-module M, there exist indecomposable pure-injective R-modules N_i such that M is elementary equivalent to $\bigoplus_{i \in I} N_i$.*

Remark

If $\varphi \in \mathrm{pp}_R^n$ and $N_i \in \text{Mod-}R$ then

$$\varphi(\oplus_i N_i) = \oplus_i \varphi(N_i).$$

Interpretation functors

A full subcategory $\mathcal{D} \subseteq \text{Mod-}R$ is called a **definable subcategory** if it is the form

$$\mathcal{D} = \{M \in \text{Mod-}R \mid \varphi_i(M) = \psi_i(M) \text{ for all } i \in I\}$$

where φ_i, ψ_i are pairs of pp-formulae indexed by I.

An embedding $i : N \to M$ is **pure** if and only if for all $L \in R\text{-mod}$, $i \otimes L : N \otimes L \to M \otimes L$ is an embedding.

Theorem

A full subcategory $\mathcal{D} \subseteq \text{Mod-}R$ is a definable subcategory if and only if $\mathcal{D}$ is closed under products, direct limits and pure-submodules.

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Interpretation functors

Definition

Let R, S be rings and $\mathcal{D}$ a definable subcategory of Mod-S.
Suppose that φ, ψ are pp-n-formulae over S and that for each $r \in R$, $\rho_r(\overline{x_1}, \overline{x_2})$ is a pp-$2n$-formula in variables $\overline{x_1}, \overline{x_2}$ each of length n.
Suppose that for all $M \in \mathcal{D}$ the following hold:

1. $\varphi(M) \supseteq \psi(M)$
2. for all $r \in R$, $\rho_r(\overline{x_1}, \overline{x_2})$ defines an endomorphism ρ_r^M of the abelian group $\varphi(M)/\psi(M)$
3. $\varphi(M)/\psi(M)$ equipped with the ρ_r^M actions is an R-module.

Then $(\varphi, \psi, (\rho_r)_{r\in R})$ defines an additive functor

$$I : \mathcal{D} \longrightarrow \text{Mod-}R, \quad M \mapsto (\varphi(M)/\psi(M), (\rho_r^M)_{r\in R}).$$

We call any such functor an **interpretation functor**.

Interpretation functors - An example

Aim: Find a definable subcategory $\mathcal{D} \subseteq \text{Mod-}kQ$ and an interpretation functor $I : \mathcal{D} \to \text{Mod-}k[T]$ such that I is essentially surjective.

Let $P_0(x) := \exists y \; ye_0 = x$ and $P_i(x) := \exists y \; y\alpha_i = x$ for $1 \leq i \leq 4$.

Let φ be $P_1(x)$ and ψ be $x = 0$.

Let $\Delta(x_2, x_1) := \exists x_3 \; x_1 + x_2 = x_3 \wedge P_1(x_1) \wedge P_2(x_2) \wedge P_3(x_3)$.

Let $\mu(x_1, x_2) := \exists x_4 \; x_1 + x_2 = x_4 \wedge P_1(x_1) \wedge P_2(x_2) \wedge P_4(x_4)$.

Let $\rho_T(x, y) := \exists z \; \mu(x, z) \wedge \Delta(z, y)$.

Let $\mathcal{D}$ be the definable subcategory

$$\{M \in \text{Mod-}kQ \mid \rho_T(x, y) \text{ defines a function from } P_1(M) \text{ to } P_1(M)\}.$$

Let R, S be rings.

Theorem (Krause, Prest)

Let $\mathcal{D}$ be a definable subcategory of $\text{Mod-}S$. *An additive functor* $I : \mathcal{D} \longrightarrow \text{Mod-}R$ *is an interpretation functor if and only if* I *commutes with direct limits and products.*

Proposition

Let $I : \text{Mod-}S \to \text{Mod-}R$ *be an interpretation functor such that* $\langle I\text{Mod-}S \rangle = \text{Mod-}R$. *There is an* $n \in \mathbb{N}$ *and a lattice embedding* $i : \text{pp}_R^1 \hookrightarrow \text{pp}_S^n$.

Remark

Let $I : \text{Mod-}S \to \text{Mod-}R$ *be an interpretation functor. If* $M \in \text{Mod-}S$ *is pure-injective then* IM *is pure-injective.*

Theorem (G.)

Let $I : \text{Mod-}S \to \text{Mod-}R$ *be an interpretation functor such that* I *maps finitely presented* S*-modules to finitely presented* R*-modules. If* I *is full on finitely presented* S*-modules then* I *is full on pure-injective* S*-modules.*