GRASSMANNIAN (CLUSTER?) CATEGORIES OF INFINITE RANK & COUNTABLE CM-TYPE

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Some parts in arXiv: 2007.14224

<u>Goal</u>: limit construction: $Gr(k,\infty) \to$ categorification

<u>Plan</u>
I Cluster algebras - grassmannians
II Cluster cats of type A + triangulations

III Grassmannian cluster cats $Gr(k,n)$
IV Infinite construction $Gr(k,\infty)$
V $k=2$: countable CM-type + triangulations

1. Cluster algebras

→ introduced by [Fomin-Zelevinsky] ~ 2000 $\quad |Q_0| = n$

Here: Cluster algs from (finite) quivers $Q = (Q_0, Q_1)$

$\mathcal{A}_Q \subseteq \mathbb{C}(x_1, \dots, x_n)$

comm. alg, get from <u>initial seed</u> $((x_1, \dots, x_n), Q)$ via mutation (some vertices may be frozen)

<u>ex</u>: $Q = 1 \to 2$ $\qquad \mathcal{A}_Q = \mathbb{C}\Big[\underbrace{x_1, x_2}_{\text{initial cluster}}, \frac{1+x_2}{x_1}, \frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1 x_2}\Big]$

Our main ex: [Scott '06] $\mathcal{A}_{k,n} := \mathbb{C}[Gr(k,n)]$ = homogeneous coord. ring of the Grassmannian of k-planes in $\mathbb{C}^n$, has structure of a cluster algebra:

$$\mathcal{A}_{k,n} = \mathbb{C}[\underbrace{p_I}_{\text{Plücker coords}} : I \subset \{1, \dots, n\}, |I| = k] \,/\, \mathcal{I}_p$$

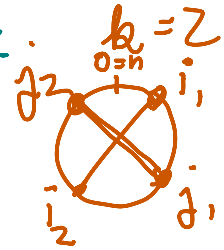
ideal of Plücker relations

$$\mathcal{I}_p = \Big\langle \sum_{r=0}^{k} (-1)^r p_{J' \cup \{j_r\}} \, p_{J \setminus \{j_r\}} \quad J, J' \subset [n],\ |J| = k+1,\ |J'| = k-1,\ J = \{j_0, \dots, j_k\}. \Big\rangle$$

What are clusters?

- max. sets of <u>compatible</u> Plücker coordinates are (examples of) clusters
- exchange relations (= mutation relations) ⟷ Plücker rels

Def $I, J \in Z$ are <u>crossing</u> if $\exists\, i_1, i_2 \in I \setminus J$ and $\exists\, j_1, j_2 \in J \setminus I$ s.t. $i_1 < j_1 < i_2 < j_2$ or



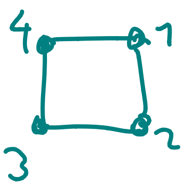
$$j_1 < i_1 < j_2 < i_2 .$$

p_I and p_J are <u>compatible</u> if I & J are noncrossing

<u>ex</u>: for $k=2$, i.e. $\mathcal{A}_{2,n}$: $p_I \xleftrightarrow{1:1}$ cluster variables

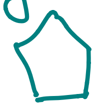
Plücker rel $\longleftrightarrow$ (mutation) exchange formulas

e.g.: $\mathcal{A}_{2,4} = \mathbb{C}[\underbrace{p_{12}, p_{23}, p_{34}, p_{14}}_{\text{frozen}}, p_{13}, p_{24}] / (p_{13}p_{24} - p_{12}p_{34} - p_{14}p_{23})$



II Cluster cats (type A_n) (ignore frozen variables)

$= Gr(2,n) \setminus$ frozen variables

ALGEBRA $\xleftarrow{\text{cc-map}}$	CATEGORY	COMBINATORICS
$\mathcal{A}_{A_{n-3}}$ $1 \to 2 \to \cdots \to n-3$	$\mathcal{C}(A_{n-3}) := D^b(\mathrm{mod}\, \mathbb{C}A_{n-3})/_{\tau^{-1}[1]}$ cluster cat: Δed, 2-CY [Buan-Marsh-Reineke-Reiten-Todorov '06]	regular n-gon  $n=5$ [Caldero-Chapoton-Schiffler]
· cluster vars	· indec objects $(\mathrm{mod}\, Q \cup \{P_i[1]\})$	· diagonals
· cluster	· cluster tilting obj $T = \bigoplus_{i=1}^{n-3} T_i$ (= max. rigid in $\mathcal{C}(A_{n-3})$, i.e. $\mathrm{Ext}^1(T_i, T_j) = 0 \ \forall i,j$)	· triangulation (= max. set of non-crossing diags)
· mutation	· T-approxim.	flip of diag

ex: $\mathcal{A}_{2,5}$: $Q = 1 \to 2$
$\mathcal{C}(1 \to 2)$: AR-quiver

Natural question: What happens for $n \to \infty$?
Can we define "$\mathcal{C}(Gr(2,\infty))$" and more generally $\mathcal{C}(Gr(k,\infty))$?

III Grassmannian cluster categories

[Jensen-King-Su '2016]: additive categorification of $\mathcal{A}_{k,n}$: $\mathcal{C}(k,n)$

$\exists$ CC-map

- rigid indec. $\overset{1-1}{\longleftrightarrow}$ cluster variables
- cluster tilting obj $\longleftrightarrow$ clusters

[$\mathcal{C}(k,n)$ is equiv. [Geiss-Leclerc-Schröer] $\mathrm{Sub}\,Q_k$]

$\mathcal{C}(k,n)$ can be described as cat. of MCM-modules:

Assume: $2 \leq k \leq \frac{n}{2}$ and $S := \mathbb{C}[x,y]$ and

$$\mu_n := \{\xi \in \mathbb{C} : \xi^n = 1\}$$

$\mu_n \curvearrowright S$ via $x \mapsto \xi x$

$y \mapsto \underset{\overset{\shortparallel}{\xi^{n-1}}}{\xi^{-1}} y$

Consider $R_{k,n} := S/(\underbrace{x^k - y^{n-k}}_{\mu_n\text{- semi invariant}})$ $\quad [\hat{R}_{k,n}]$

$t = xy$ μ_n-inv.

$\mathcal{C}(k,n) := \mathrm{MCM}_{\mu_n}(R_{k,n}) = \mathrm{MCM}(\underbrace{R_{k,n} * \mu_n}_{\text{A skew group ring}})$

$\mathcal{C}(k,n)$ is Frobenius, $\underline{\mathcal{C}}(k,n)$ is Hom-finite, 2-CY
Moreover:
{rank 1 mod. over A} $\overset{1-1}{\longleftrightarrow}$ {Plücker coords}

- {cluster tilting object} $\longleftarrow$ {max. set of compat. Plücker}

IV ∞-construction $Gr(k,\infty)$

Fix $k \in \mathbb{Z}_{\geq 2}$, "$n \to \infty$"

Algebra: [Grabowski-Gratz; Groechenig]
[Gratz] $A_k = \mathbb{C}[p_I : I \subset \mathbb{Z}, |I| = k] / J_p$
can be endowed with a cluster structure

Construction of $\mathcal{C}(k,\infty)$:
[JKS] + "$n \to \infty$": $\mu_n \to G_m = \mathbb{C}^*$ acts on $S = \mathbb{C}[x,y]$
via $x \mapsto \xi x$
$y \mapsto \xi^{-1} y$ $\quad \xi \in \mathbb{C}^*$

$R_k := \mathbb{C}[x,y]/(\underbrace{x^k}_{x^k - y^{n-k} \to 0})$

Define: $MCM_{G_m}(R_k) =:$ Grassmannian cat of ∞-rank

Character grp of $\mathbb{G}_m$ is $\mathbb{Z}$:
$$\mathrm{mod}_{\mathbb{G}_m} R_k \simeq \mathrm{mod}_{\mathbb{Z}} R_k \qquad |x| = 1,\ |y| = -1$$

$$\Rightarrow MCM_{\mathbb{G}_m}(R_k) \simeq MCM_{\mathbb{Z}}(\underset{\mathbb{C}[x,y]/(x^k)}{\underset{\shortparallel}{R_k}})$$

Good news: • R_k is hypersurface $\Rightarrow$ MCM's can be described by matrix factorizations (MF)

• If $k=2$, then $\mathbb{C}[x,y]/(x^2)$ is of countable CM-type + MF are classified [Buchweitz–Greuel–Schreyer 1987]

"type A_∞"

• cluster combinatorics of type A_∞:
[Holm–Jørgensen]
[Igusa–Todorov]
↓
[Paquette–Yildirim] '20

Bad news: $k \geq 3$: R_k is of wild CM-type

Def Let $\mathcal{F} = (R_k)_{(x)}$

$M \in MCM_{\mathbb{Z}}(R_k)$ is <u>generically free</u> of rk n if $M \otimes_{R_k} \mathcal{F}$ is a free rank n $\mathcal{F}$-module.

$MCM^0_{\mathbb{Z}}(R_k)$ = full subcat. of gen. free MCM R_k-mod.

Prop [ACFGS]:

(1) If $M \in MCM^0_{\mathbb{Z}}(R_k)$, then $M = Syz(N)$, $N \in mod_{\mathbb{Z}} R_k$ of finite length

(2) $M \in MCM^0_{\mathbb{Z}}(R_k)$, $rk(M) = 1 \iff M \cong$ to graded ideal of R_k and $y^n \in M$, $n > 0$.

(3) Every hom. ideal in R_k can be generated by monomials.

Thm [ACFGS]: $I \in MCM^0_{\mathbb{Z}}(R_k)$, $rk(I) = 1 \iff$

$I = (x^{k-1}, x^{k-2}y^{i_1}, x^{k-3}y^{i_2}, \dots, xy^{i_{k-2}}, y^{i_{k-1}})(i_k)$

$0 \leq i_1 \leq \dots \leq i_{k-1}$, $i_k \in \mathbb{Z}$

Def: $\ell(I) = (-i_{k-1} - i_k, -i_{k-2} - i_k + 1, \dots, -i_k + k - 1)$ $\quad -i_{k-3} - i_k + 2$

strictly increasing k-tuple

Cor $\{$gen. free rk 1, $MCM_{\mathbb{Z}}$-mod$\} \xleftrightarrow{1-1} \{$Plücker coords of $\mathcal{U}_k\}$

$I \longmapsto P_{e(I)}$

Thm [ACFGS] If $I, J \in MCM^0_{\mathbb{Z}}(R_k)$, rk 1, then $Ext^1(I,J)=0 \iff P_{e(I)}$ and $P_{e(J)}$ are compatible.

Cor: $Ext^1(I,J) = Ext^1(J,I)$ and I is rigid.

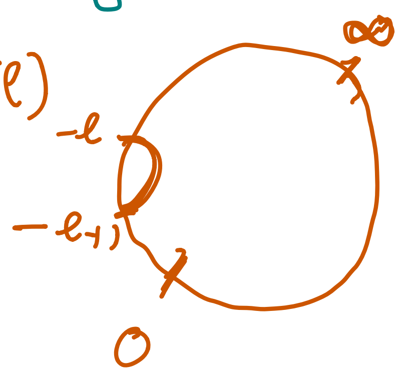
$MCM^0_{\mathbb{Z}}(R_k)$ is stably 2-CY [using Iyama-Takahashi]

V $k=2$: $\mathbb{C}[x,y]/(x^2)$

Thm [BGS]: $\mathbb{C}[x,y]/(f)$, $f \neq 0$ is countable CM-type (not finite)

$\iff \underbrace{f = x^2}_{A_\infty}$ or $\underbrace{f = x^2 y}_{D_\infty}$

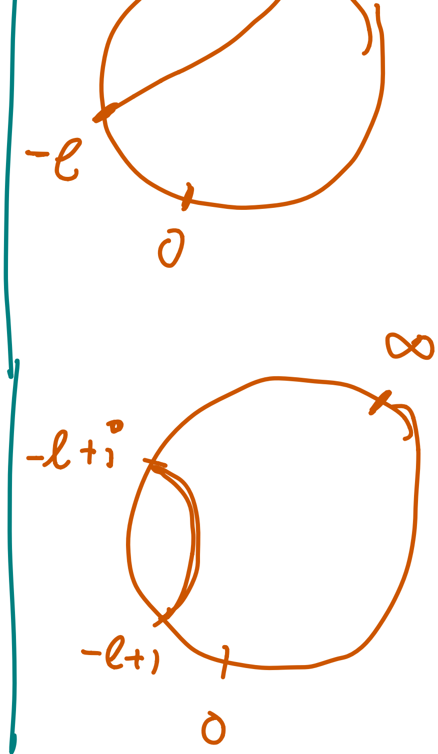
A_∞

MF	MCM	∞-gon [ACFGS]
$(x^2,1)(\ell)$ $S \xrightarrow{1} S \xrightarrow{x^2} S$	$\mathrm{coker}(x^2) = R = \mathbb{C}[x,y]/(x^2)(\ell)$	
$(x,x)(\ell)$ $S \xrightarrow{x} S \xrightarrow{x} S$	$\mathrm{coker}(x) = \mathbb{C}[y](\ell)$	

$\left[\begin{pmatrix} x & y^i \\ 0 & -x \end{pmatrix}, \begin{pmatrix} x & y^i \\ 0 & -x \end{pmatrix}\right]$

$i = 1, 2, \ldots$

$\operatorname{coker} \cong (x, y^i)(\ell)$



<u>Theorem</u>: cluster tilting subcat.

"Double fountain"

max. rigid: