On Regular and (Barr-)Exact Categories: ordinary and poset-enriched

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NEU Representation Theory and Related Topics Seminar

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Outline

Regular Categories Naively

1. Non-additive version of abelian category (exactness even closer though)
2. Good notion of image of a morphism.
3. Good calculus of internal relations.
4. The fragment of first-order logic on $\wedge, \top, \exists$.

Kernel Pairs - Coequalizers

▶Given $f : X \to Y$ in a category $\mathcal{C}$, the *kernel pair* of f is the pullback

$$\begin{array}{ccc} K(f) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ X & \xrightarrow[f]{} & Y \end{array}$$

In terms of elements: $K(f) = \{(x, x') \in X \times X | f(x) = f(x')\}$

▶The *coequalizer* of a pair $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is the colimit

$X \underset{g}{\overset{f}{\rightrightarrows}} Y \overset{q}{\twoheadrightarrow} Q$, i.e. $qf = qg$ and universality:
$hf = fg \implies (\exists! u : Q \to Z) uq = h.$

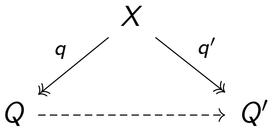
If kernel pairs and their coequalizers exist in a category $\mathcal{C}$, then for every $X \in \mathcal{C}$

$$\mathrm{Eff(X)} \underset{\longleftarrow}{\overset{\longrightarrow}{\cong}} \mathrm{Quot(X)}$$

where

▶ $\mathrm{Eff(X)}$ is the poset of *effective equivalence relations* on X (=kernel pairs of morphisms $f : X \to Y$).

▶ $\mathrm{Quot(X)}$ is the poset of *regular* epis $q : X \twoheadrightarrow Q$ (=coequalizers of some pair of parallel arrows)



$q' \leq q$

(One) Definition of Regularity

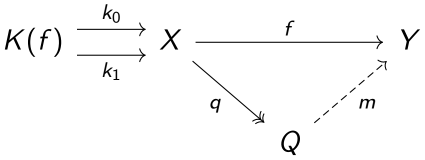
Definition

A finitely complete category $\mathcal{C}$ is called *regular* if

1. $\mathcal{C}$ has coequalizers of kernel pairs.
2. Regular epimorphisms are stable under pullback in $\mathcal{C}$.

Constructing Factorizations

Suppose that $\mathcal{C}$ is regular and let $f : X \to Y \in \mathcal{C}$. We can then consider the following diagram:



where $q = \mathrm{coeq}(\mathrm{k}_0, \mathrm{k}_1)$.

Stability of regular epis $\implies$ m is a mono

Regularity = Image Factorization

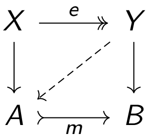
Proposition

A finitely complete category $\mathcal{C}$ is regular iff

1. *$\mathcal{C}$ has (regular epi, mono) factorizations.*
2. *Regular epimorphisms are stable under pullback in $\mathcal{C}$.*

Also important: In a regular category $\mathcal{C}$: regular epis=strong epis

$e : X \to Y$ is a *strong* epi if for all monos $m : A \rightarrowtail B$



$e \perp m$

Examples of Regular Categories

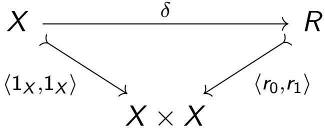
1. Set, Set_{fin}, any elementary topos.
2. Any abelian category.
3. Every *quasi-variety* of Universal Algebra, e.g. $\mathsf{Ab}_{t.f.}$, Ring_{red}
4. Every $[\mathcal{C}, \mathcal{E}]$, for regular $\mathcal{E}$.
5. $\mathsf{Set}^{\mathbb{T}}$ for any monad $\mathbb{T}$. e.g. CHaus, G-Set
6. Stone   (*Stone Duality:* Stone $\simeq$ Bool^{op})
7. Grp(Top), $\mathbb{T}$(Top) for any sufficiently nice algebraic theory $\mathbb{T}$ (e.g. Mal'tsev theories)

Non-examples: Pos, Top, Cat

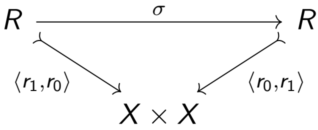
Barr-exactness

Let $\mathcal{C}$ be a finitely complete category. A relation $\langle r_0, r_1\rangle : R \rightarrowtail X \times X$ on an object $X \in \mathcal{C}$ is:

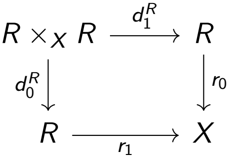
reflexive



symmetric



transitive

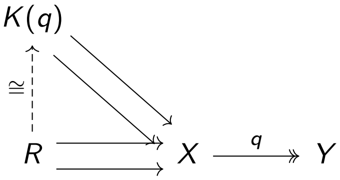


$(\exists \tau : R \times_X R \to R)\ r_1\tau = r_1 d_1^R\ ,\ r_0\tau = r_0 d_0^R.$

Definition (M. Barr, 1971)

A category $\mathcal{E}$ is called *exact* if it is regular and every equivalence relation in $\mathcal{E}$ is effective.

Equivalently, $\mathcal{E}$ is regular and for every equivalence relation $R \rightarrowtail X \times X \in \mathcal{E}$:

1. R has a coequalizer $q : X \twoheadrightarrow Y$.
2. 

Examples of Exact Categories

1. Set and any presheaf category $[\mathcal{C}, \mathrm{Set}]$ are exact. More generally, every elementary topos is exact.
2. Every abelian category.
3. Every *variety* of universal algebras, e.g. Grp, Mon, Ring.
4. $\mathrm{Set}^{\mathbb{T}}$, for any monad $\mathbb{T}$ on Set. Examples: CHaus, G-Set

Non-examples: $\mathrm{Ab}_{t.f.}$, Mon_{can}, Ring_{red}, Stone, Grp(Top)

Why is $\mathsf{Ab}_{t.f.}$ not exact?

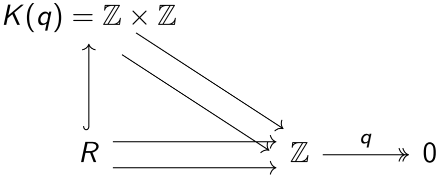
We mentioned that $\mathsf{Ab}_{t.f.}$ is regular.

It also has coequalizers of equivalence relations. In fact, all colimits.

But consider the relation R on $\mathbb{Z}$ defined by

$$(m, n) \in R \iff m - n = 2k$$

Then R is an internal equivalence relation which is not effective



Exactness + Additivity

In fact, in the additive context exactness characterizes abelian categories:

Theorem (M. Tierney, ∼1970)

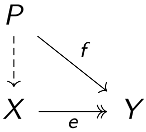
A category $\mathcal{A}$ is abelian iff it is additive and (Barr-)exact.

▶ Basic Idea: In an additive category, every reflexive relation is an equivalence relation (*Mal'tsev* property)

Characterizing (Quasi-)varieties

An object P in a regular category $\mathcal{E}$ is:

- *(regular) projective* if for every regular epi $e : X \twoheadrightarrow Y \in \mathcal{E}$



- a *(regular) generator* if $\mathcal{E}(P, X) \bullet P \twoheadrightarrow X$ is a regular epi $\forall X \in \mathcal{E}$.
- *finitely presentable* if $\mathcal{E}(P, -) : \mathcal{E} \to \mathrm{Set}$ preserves filtered colimits.

Theorem (F.W. Lawvere, 1963)

A category $\mathcal{E}$ is equivalent to a variety iff it is exact and has an object P which is a projective, finitely presentable, regular generator.

Theorem (J. Duskin, 1969)

A category $\mathcal{E}$ is equivalent to one of the form $\mathrm{Set}^{\mathbb{T}}$ for a monad $\mathbb{T}$ iff it is exact and has a projective regular generator P.

In both cases $P = F(1)$, the free algebra on the singleton.

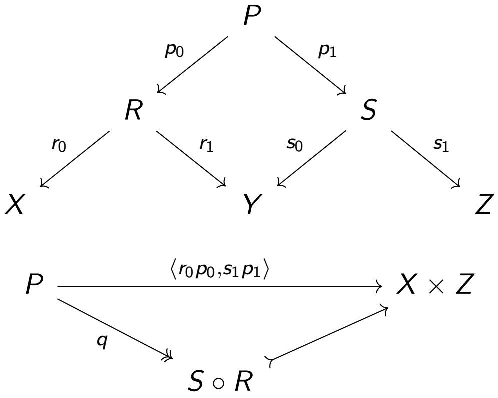
Theorem

A category $\mathcal{C}$ is equivalent to a quasi-variety iff it is regular, has a projective, finitely presentable, regular generator and admits coequalizers of equivalence relations.

Composing Relations in a Regular Category

A relation $R : X \looparrowright Y$ is a subobject $R \rightarrowtail X \times Y$. If $\mathcal{E}$ has finite limits and image factorizations, for any $R : X \looparrowright Y$ and $S : Y \looparrowright Z$ we can define $S \circ R : X \looparrowright Z$ as follows:

First construct the pullback square in the center of the diagram below



The bicategory of relations

Then regularity of $\mathcal{E}$ implies the following

Proposition

For any $R : X \looparrowright Y$, $S : Y \looparrowright Z$ *and* $T : Z \looparrowright W$ *in a regular category* $\mathcal{E}$

$$T \circ (S \circ R) = (T \circ S) \circ R$$

We thus have the *bicategory of relations* $\mathrm{Rel}(\mathcal{E})$. Identity morphisms are $\Delta_X = \langle 1_X, 1_X \rangle : X \rightarrowtail X \times X$.
Every morphism $f : X \to Y \in \mathcal{E}$ defines a relation $f : X \looparrowright Y$ via its *graph* $\langle 1_X, f \rangle : X \rightarrowtail X \times Y$. We thus have a faithful functor $\mathcal{E} \to \mathrm{Rel}(\mathcal{E})$.

The Structure of $\mathrm{Rel}(\mathcal{E})$

The locally posetal bicategory $\mathrm{Rel}(\mathcal{E})$ has:

- Binary $\cap$ and $\top$.
- An *involution* $(-)^\circ : \mathrm{Rel}(\mathcal{E})^{\mathrm{op}} \to \mathrm{Rel}(\mathcal{E})$ which preserves $\subseteq$.
- Freyd's *Modular Law*

$$SR \cap T \subseteq S(R \cap S^\circ T)$$

Terminology: $\mathrm{Rel}(\mathcal{E})$ is an *allegory*.

Maps in $\mathrm{Rel}(\mathcal{E})$

For every $f : X \to Y \in \mathcal{E}$ we have:

- $f^\circ f \supseteq \Delta_X$, (in fact, $f^\circ f = K(f)$)
- $ff^\circ \subseteq \Delta_Y$

i.e. $f \dashv f^\circ$ in $\mathrm{Rel}(\mathcal{E})$. Every $f : X \to Y \in \mathcal{E}$ is a *map* in $\mathrm{Rel}(\mathcal{E})$.
Actually,

Proposition

If $R : X \nrightarrow Y$ is a map in $\mathrm{Rel}(\mathcal{E})$, i.e. there exists $S : Y \nrightarrow X$ such that $R \dashv S$, then there is an $f : X \to Y \in \mathcal{E}$ with $R = f$.

Compactly: For any regular $\mathcal{E}$, $\mathcal{E} \simeq \mathrm{Map}(\mathrm{Rel}(\mathcal{E}))$.

(Co)Limit properties via Relations

Consider a square in the regular category $\mathcal{E}$.

$$\begin{array}{ccc} P & \xrightarrow{q} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

Then the square is:

- Commutative iff $qp^\circ \subseteq g^\circ f$
- Commutative and s.t. $P \to X \times_Z Y$ is a regular epi iff $qp^\circ = g^\circ f$
- a pullback iff $qp^\circ = g^\circ f$ and $p^\circ p \cap q^\circ q = \Delta_P$.

Exactness of $\mathcal{E}$ via $\mathrm{Rel}(\mathcal{E})$

When $E : X \looparrowright X$ is an effective equivalence relation in $\mathcal{E}$, then in $\mathrm{Rel}(\mathcal{E})$ we have $E = q^\circ q$ and $qq^\circ = \Delta_Y$, where q is the coequalizer of E.

Translation: E effective $\implies$ E is a *split idempotent* in $\mathrm{Rel}(\mathcal{E})$.

Proposition

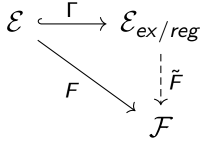
An equivalence relation E in $\mathcal{E}$ is effective iff it is a split idempotent in $\mathrm{Rel}(\mathcal{E})$.

Proof.

Assume $E = SR$ and $RS = \Delta_Y$ for some $R : X \looparrowright Y$ and $S : Y \looparrowright X$. Then $SR = E \supseteq \Delta_X$ and $RS \subseteq \Delta_Y$, so there is an $f : X \to Y \in \mathcal{E}$ such that $R = f$, $S = f^\circ$. Now $E = SR = f^\circ f = K(f)$. □

The Exact Completion

Given the regular category $\mathcal{E}$, we can construct a *free* exact category $\mathcal{E}_{ex/reg}$



for every regular $F : \mathcal{E} \to \mathcal{F}$ with $\mathcal{F}$ exact.

▶Split the equivalence relations as idempotents in $\mathrm{Rel}(\mathcal{E})$. (This yields $\mathrm{Rel}(\mathcal{E}_{\mathrm{ex/reg}})$)

▶Take maps in the resulting bicategory.

$\mathcal{E}_{ex/reg}$ concretely

- Objects: Pairs (X, E) where E is an equivalence relation on the object X in $\mathcal{E}$.
- Morphisms: A morphism $R : (X, E) \to (Y, G)$ is a relation $R : X \looparrowright Y$ in $\mathcal{E}$ such that
  1. $GRE = R$.
  2. $R^{\circ} R \supseteq E$.
  3. $RR^{\circ} \subseteq G$.

$\Gamma : \mathcal{E} \to \mathcal{E}_{ex/reg}$ maps X to (X, Δ_X) and for every $(X, E) \in \mathcal{E}_{ex/reg}$ there is an *exact sequence*

$$\Gamma(E) \underset{\Gamma e_1}{\overset{\Gamma e_0}{\rightrightarrows}} \Gamma X \overset{E}{\twoheadrightarrow} (X, E)$$

A few examples

- $(\mathrm{Ab}_{t.f.})_{ex/reg} \simeq \mathrm{Ab}$
- $(\mathrm{CRing}_{red})_{ex/reg} \simeq \mathrm{CRing}$
- $\mathrm{Stone}_{ex/reg} \simeq \mathrm{CHaus}$

Shameless plug: These can also be deduced from a joint result with P. Karazeris (TAC, 2017) which implies that $\Gamma : \mathcal{E} \to \mathcal{E}_{ex/reg}$ is the unique fully faithful, regular $F : \mathcal{E} \to \mathcal{F}$ with $\mathcal{F}$ exact and such that

$$(\forall Y \in \mathcal{F}) \;\exists\, FX \twoheadrightarrow Y$$

Sample Application

We can use the ex/reg completion to characterize (nice) reflections of classes of exact categories.

Simplest case: A regular category $\mathcal{E}$ is a reflective subcategory of an exact category iff it has coequalizers of equivalence relations.
If $\mathcal{E}$ has such coequalizers, then $\Gamma : \mathcal{E} \to \mathcal{E}_{ex/reg}$ has a left adjoint $L : \mathcal{E}_{ex/reg} \to \mathcal{E}$

$$E \rightrightarrows X \twoheadrightarrow L(X, E)$$

Next:(S. Mantovani) The coequalizers in $\mathcal{E}$ of equivalence relations are stable iff $L : \mathcal{E}_{ex/reg} \to \mathcal{E}$ is a *semi-localization*.

Corollary

$\mathcal{F}$ is a torsion-free class in an abelian category iff $\mathcal{F}$ is regular, additive and has stable coequalizers of equivalence relations.

$\mathcal{F}$ regular additive $\implies$ $\mathcal{F}_{ex/reg}$ additive exact (=abelian)

Pos-enriched Categories

We are interested in Pos-enriched categories, i.e.

Categories $\mathcal{C}$ such that:

- $\mathrm{Hom}_{\mathcal{C}}(\mathrm{X}, \mathrm{Y})$ is a poset for all $X, Y \in \mathcal{C}$.
- For $f, f' : X \to Y$, $f \leq f' \implies hfg \leq hf'g$.

e.g. Ordered (quasi-)varieties, distributive lattices, Priestley spaces, Compact Ordered spaces (Nachbin)

<u>Motivation</u>:*Quasivarieties and varieties of ordered algebras: regularity and exactness*. A. Kurz, J. Velebil, Math. Struct. in Comp. Science (2017)

Pos-enriched Regularity

Regularity for Pos-enriched categories involves:

Kernel congruences: For any $f : X \to Y$, comma square

$$\begin{array}{ccc} f/f & \longrightarrow & X \\ \downarrow & \leq & \downarrow f \\ X & \xrightarrow[f]{} & Y \end{array}$$

In terms of elements, $f/f = \{(x, x')|fx \leq fx'\}$.

Coinserters: $X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \xrightarrow{q} Q$, $\quad qf_0 \leq qf_1$ and q is universal with this property.

ff-morphism (*order-mono*): m such that $mf \leq mg \implies f \leq g$

A Pos-category $\mathcal{C}$ is *regular* if

1. $\mathcal{C}$ has finite weighted limits.
2. $\mathcal{C}$ has stable under pullback (coinserter, ff) factorizations.

Pos-enriched Exactness

Exactness involves *congruences*, i.e. internal relations $E \rightarrowtail X \times X$ such that:

1. E is *reflexive* and *transitive*.
2. E is *weakening-closed*, i.e. it satisfies

$$x' \leq x, \;\; (x, y) \in E, \;\; y \leq y' \implies (x', y') \in E.$$

$\mathcal{C}$ is *exact* if every congruence is f/f for some $f : X \to Y$.

$\mathrm{Rel}(\mathcal{E})$ is still good for calculations, BUT cannot recover morphisms of $\mathcal{E}$ as maps.

To capture the enriched nature of $\mathcal{E}$ we must consider the bicategory $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ of weakening-closed relations.

$$
\begin{array}{ccc}
f_* & \longrightarrow & Y \\
\big\downarrow & \leq & \big\downarrow 1_Y \\
X & \xrightarrow[f]{} & Y
\end{array}
\qquad
\begin{array}{ccc}
f^* & \longrightarrow & X \\
\big\downarrow & \leq & \big\downarrow f \\
Y & \xrightarrow[1_Y]{} & Y
\end{array}
\qquad f_* \dashv f^* \text{ in } \mathrm{Rel}_{\mathrm{w}}(\mathcal{E}).
$$

$$f_* = \{(x, y) \in X \times Y | f(x) \leq y\} \quad f^* = \{(y, x) \in Y \times X | y \leq f(x)\}$$

Proposition

If $R : X \looparrowright Y$ and $S : Y \looparrowright X$ in $\mathrm{Rel}_{\mathrm{w}}(\mathcal{E})$ are such that $R \dashv S$, then $R = f_*$ and $S = f^*$ for some $f : X \to Y \in \mathcal{E}$.

Pos-enriched Exact Completion

We would like to construct $\mathcal{E}_{ex/reg}$ in this context by performing a similar construction as in the ordinary case.

HOWEVER, $\mathrm{Rel_w}(\mathcal{E})$ doesn't have as rich a structure as $\mathrm{Rel}(\mathcal{E})$ (no symmetry!) and is also not sufficient for formulating all limit/exactness properties.
Nevertheless, we have shown that we can make it work: split idempotents in both $\mathrm{Rel}(\mathcal{E})$ and $\mathrm{Rel_w}(\mathcal{E})$ at the same time, take maps in the second one.

One interesting example: $\mathsf{Pries}_{ex/reg} \simeq \mathsf{Nach}$ ($\simeq \mathsf{Stone}_{ex/reg}$).

▶Preprint: *An Exact Completion for Regular Categories Enriched in Posets*

Thank you for your attention!

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