Gorenstein transpose and its dual

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Contents

Introduction

1. Transpose (see [AB])
Let R be a left and right Northerian ring. For $M \in \operatorname{mod} R$, there exists a projective presentation in mod R:

$$P_1 \xrightarrow{f} P_0 \to M \to 0.$$

Then we get an exact sequence in mod R^{op}:

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{coker} f^* \to 0,$$

where $()^* = \operatorname{Hom}(-, R)$. $\operatorname{coker} f^*$ is called a *transpose* of M, and denoted by $\operatorname{Tr} M$.

Remark. The transpose of M depends on the choice of the projective presentation of M, but it is unique up to projective equivalence.

Introduction

Gorenstein projective module

Auslander's original definition([AB]):

Let R be a left and right Noetherian ring and M a finitely generated R-module. Recall that M has G-dimension 0 (or G-dim$M = 0$) if $\mathrm{Ext}^i(M,R) = 0 = \mathrm{Ext}^i(M^*,R)$ for $i > 0$ and M is reflexive (i.e. $M \cong M^{**}$)

Introduction

- A left R-module $M \in \text{Mod}\, R$ is called *Gorenstein projective* [EJ] if there is an exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective left R-modules with $M = \mathrm{coker}(P_1 \to P_0)$ and such that $\mathrm{Hom}(-, P)$ leaves the sequence exact for each projective left R-module P. Denote the class of all Gorenstein projective left R-modules by $\mathcal{GP}(R)$.

- A *f.g.* module M over a Noetherian ring is Gorenstein projective iff G-dim$M = 0$.

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- A *f.g.* module M over a Noetherian ring is Gorenstein projective iff $G\text{-dim}M = 0$.

Introduction

- A left R-module M is called *Gorenstein injective* if there is an exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of injective left R-modules with $M = \ker(E^0 \to E^1)$ such that $\mathrm{Hom}(E, -)$ leaves the sequence exact whenever E is an injective left R-module.

Introduction

2. Gorenstein Transpose (see [HH])
Let R be a left and right Northerian ring. For $M \in \text{mod } R$, there exists a Gorenstein projective presentation in mod R:

$$G_1 \xrightarrow{g} G_0 \to M \to 0.$$

Then we get an exact sequence in mod R^{op}:

$$0 \to M^* \to G_0^* \xrightarrow{g^*} G_1^* \to \text{coker} g^* \to 0,$$

where $()^* = \text{Hom}(, R)$. $\text{coker} g^*$ is called a *Gorenstein transpose* of M, and denoted by $\text{Tr}_{\text{G}} M$.
Question. The Gorenstein transpose of M depends on the choice of the Gorenstein projective presentation of M, is it unique up to Gorenstein projective equivalence?

Motivation

Main results about the Gorenstein transpose

1. Establish a relation between a Gorenstein transpose of a module with a transpose of the same module.

[HH, Theorem 3.1]

Let $M \in \text{mod } R$ and $A \in \text{mod } R^{op}$. Then A is a Gorenstein transpose of M if and only if there exists an exact sequence $0 \to A \to \text{Tr}M \to G \to 0$ in mod R^{op} with G Gorenstein projective.

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2. Provide a method to construct a Gorenstein transpose of a module from a transpose of the same module.

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Let G be a Gorenstein projective module. Then $\mathrm{Tr}M \oplus G$ is a Gorenstein transpose of M.

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However, the following two questions remain unknown:

- Is it true that the Gorenstein transpose of a module is unique up to Gorenstein projective equivalence?
- Is it true that any Gorenstein transpose of a module can be obtained by directed sums of a transpose of the same module and a Gorenstein projective module?

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Our main results

Idea Note that in the proof of [HH, Theorem 3.1], they mainly used the relation between Gorenstein projective modules and projective modules (generator).

[ZS, Theorem 2.1]

Let R be a left and right Noetherian ring, $M \in \mathrm{mod}\ R$. Then $\mathrm{Tr}_G M$ is a transpose of N, where $N \in \mathrm{Ext}(\mathcal{GP}(R), M)$ is an extension of a Gorenstein projective R-module by M, which means that there is an exact sequence $0 \to M \to N \to G \to 0$ in mod R with G Gorenstein projective.

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Our main results

[ZS, Theorem 2.3]

Suppose that $M \in \text{mod } R$. Then, for any Gorenstein transpose of M, there exists an exact sequence $0 \to H \to \mathrm{Tr}_G M \to \mathrm{Tr} M \to 0$ in mod R^{op} with H Gorenstein projective.

Remark. We do not know whether the converse is true.

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Our main results

[ZS, Corollary 2.4]

Any two Gorenstein transposes of $M \in \text{mod } R$ are Gorenstein projectively equivalent.

[ZS, Corollary 2.5]

If $M \in \text{mod } R$ has finite projective dimension, then, for any Gorenstein transpose of M, there is a Gorenstein projective modules G, such that $\mathrm{Tr}_G M = \mathrm{Tr} M \oplus G$.

The dual of the transpose

Let R and S be associative rings with units. We use Mod R (resp. Mod S^{op}) to denote the class of left R-modules (resp. right S-modules).

Definition ([HW])

An (R, S)-bimodule $C = {}_RC_S$ is called *semidualizing* if it satisfies the following.
(a) ${}_RC$ and C_S admit a resolution by finitely generated projective left R-modules and projective right S-modules, respectively.
(b) The maps $R \to \mathrm{Hom}_{S^{op}}(C, C)$ and $S \to \mathrm{Hom}_R(C, C)$ are isomorphisms.
(c) $\mathrm{Ext}_R^{i\geqslant 1}(C, C) = 0 = \mathrm{Ext}_{S^{op}}^{i\geqslant 1}(C, C)$.

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Let $M \in \text{Mod}\, R$, and $0 \to M \to I^0 \xrightarrow{g} I^1$ be an injective resolution of M. We denote either $\text{Hom}_R({}_RC_S, -)$ or $\text{Hom}_{S^{op}}({}_RC_S, -)$ by $(\ )_*$. So we get an exact sequence in Mod S:

$$0 \to M_* \to I^0_* \xrightarrow{g_*} I^1_* \to \text{coker} g_* \to 0.$$

$\text{coker} g_*$ is called *cotranspose* of M with respect to C, and denoted by $\text{cTr}M$.

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- How to dualize the Gorenstein transpose of modules appropriately? Replacing an injective resolution of M by a Gorenstein injective resolution of M? (Unfortunately)
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$\mathcal{Y}$-cotranspose

Let $\mathcal{Y}$ be a subcategory of Mod R, and $U = {}_RU_S$ be a fixed (R, S)-bimodule. For convenience, we denote either $\mathrm{Hom}_R({}_RU_S, -)$ or $\mathrm{Hom}_{S^{op}}({}_RU_S, -)$ by $(\ )_*$.

Definition ([Z])

Suppose that A has an $\mathcal{Y}$-copresentation, that is, there exists an exact sequence $0 \to A \to Y^0 \xrightarrow{g} Y^1$ in Mod R with Y^0, Y^1 $\in \mathcal{Y}$. Applying the functor $(\ )_*$ to the sequence above induces an exact sequence in Mod S:

$$0 \to A_* \to Y^0_* \xrightarrow{g_*} Y^1_* \to \mathrm{coker} g_* \to 0.$$

We call $\mathrm{coker} g_*$ a *$\mathcal{Y}$-cotranspose* of A with respect to U, and denoted by $\mathrm{cTr}^U_{\mathcal{Y}}A$.

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First relation

Let $\mathcal{W}$ be a generator or cogenerator for $\mathcal{Y}$, we want to investigate the relations between $\mathcal{Y}$-cotranspose and $\mathcal{W}$-cotranspose.

[Z, Theorem 4.3]

Let $A \in \text{Mod } R$ and $\mathcal{W}$ be a cogenerator for $\mathcal{Y}$. Assume that $\mathcal{Y}$ is closed under extensions and $\text{Ext}_R^1(U, \mathcal{Y}) = 0$.
(1) If M is a $\mathcal{Y}$-cotranspose of A with respect to U, then there is an exact sequence $0 \to M \to \text{cTr}_{\mathcal{W}}^U A \to Y_* \to 0$ in Mod S with $\text{cTr}_{\mathcal{W}}^U A$ a $\mathcal{W}$-cotranspose of A and $Y \in \mathcal{Y}$.
(2) If $\mathcal{Y}$ is U-coflexive, $\mathcal{Y}_*$ is closed under kernel of epimorphism and $\text{Tor}_1^S(U, \mathcal{Y}_*) = 0$, then the converse of (1) is true.

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The definition of U-coreflexive

For a left R-module A, let $\theta_A : U \otimes_S A_* \to A$ via $\theta_A(x \otimes f) = f(x)$, for any $x \in U$ and $f \in A_*$, be the canonical evaluation homomorphism.

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A is called *U-coreflexive* if θ_A is an isomorphism.

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A is called U-*coreflexive* if θ_A is an isomorphism.

$\mathcal{L}_C$-Gorenstein injective module

Motivated by [Z, Theorem 4.3], we introduce the following notion:

Definition

A left R-module M is called $\mathcal{L}_C$-Gorenstein injective if there exists an exact sequence:

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

in $\mathcal{I}(R)$, such that $M \cong \mathrm{im}(I_0 \to I^0)$ and the sequence is $\mathrm{Hom}_R(\mathcal{I}(R), -)$-exact and $\mathrm{Hom}_R(C, -)$-exact.
Denote the class of all $\mathcal{L}_C$-Gorenstein injective left R-modules by $\mathcal{L}_C(R)$.

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$\mathcal{L}_C$-Gorenstein injective module

Fact

(1) $\mathcal{L}_C(R) = \mathcal{GI}(R) \cap \mathcal{B}_C(R)$.

(2) There are Foxby equivalences of categories:

$$
\begin{array}{ccc}
\mathcal{I}_C(S) & \underset{\operatorname{Hom}_S(C,-)}{\overset{C\otimes_R -}{\underset{\sim}{\rightleftarrows}}} & \mathcal{I}(R) \\
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Definition [HW]

The Bass class $\mathcal{B}_C(R)$ with respect to C is the subcategory of left R-modules M satisfying:
(1) $\mathrm{Ext}_R^{i\geqslant 1}(C,M)=0=\mathrm{Tor}^S_{i\geqslant 1}(C,\mathrm{Hom}_R(C,M))$ and
(2) The natural evaluation map $\theta_M : C\otimes_S \mathrm{Hom}_R(C,M)\rightarrow M$ is an isomorphism.

The Auslander class $\mathcal{A}_C(S)$ with respect to C is the subcategory of left S-modules N satisfying:
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Fact

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To give the dual counterparts of Gorenstein transposes. Motivated by Theorem 4.3, we should choose the $\mathcal{L}_C$-Gorenstein injective copresentation instead of the Gorenstein injective copresentation.

Definition

Let $A \in \text{Mod } R$. Then there exists an exact sequence $0 \to A \to G^0 \xrightarrow{g} G^1$ in Mod R with $G^0, G^1 \in \mathcal{L}_C(R)$. Applying the functor $(\ )_* = \text{Hom}_R({}_RC_S, -)$ to the sequence above induces an exact sequence in Mod S:

$$0 \to A_* \to G^0_* \xrightarrow{g_*} G^1_* \to \text{cokerg}_* \to 0.$$

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The following result can be regarded as a dual of [HH, Theorem 3.1].

[Z, Theorem 4.5]

Let $A \in \text{Mod } R$. Then any $\mathcal{L}_C$-Gorenstein cotranspose of A can be embedded into a cotranspose of A with the cokernel in $\mathcal{G}(\mathcal{I}_C(S))$.

Remark. We do not know whether the converse is true. But we have

[Z, Theorem 4.8]

Let $A \in \text{Mod } R$ and $M \in \text{Mod } S$. Then M is a $\mathcal{B}_C(R)$-cotranspose of A with respect to C if and only if there is an exact sequence: $0 \to M \to \text{cTr}A \to L \to 0$ with $L \in \mathcal{A}_C(S)$.

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Second relation

[Z, Theorem 4.11]

Let $A \in \operatorname{Mod} R$. Assume that $\mathcal{V}$ is a generator for $\mathcal{Y}$, and $\mathcal{Y}$ is closed under extensions. If $\operatorname{Ext}^1_R(U, \mathcal{Y}) = 0$, then, for any $\mathcal{Y}$-cotranspose $\operatorname{cTr}^U_{\mathcal{Y}} A$ of A, there is an isomorphism $\operatorname{cTr}^U_{\mathcal{Y}} A \cong \operatorname{cTr}^U_{\mathcal{V}} B$ for some $B \in \operatorname{Ext}(A, \mathcal{Y})$.

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[Corollary]

Any $\mathcal{L}_C$-Gorenstein cotranspose of A, is a cotranspose of B, where $B \in \mathrm{Ext}(A, \mathcal{L}_C(R))$.

This can be regarded as a dual of [ZS, Theorem 2.1].

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By Theorem 4.11, we give another relation between a $\mathcal{L}_C$-Gorenstein cotranspose of a module and a cotranspose of the same module.

[Z, Theorem 4.12]

Suppose that $A \in \text{Mod } R$. Then, for any $\mathcal{L}_C$-Gorenstein cotranspose of A, there exists an exact sequence $0 \to G \to \text{cTr}_{\mathcal{L}_C} A \oplus E \to \text{cTr} A \to 0$ in Mod S with $E \in \mathcal{I}_C(S)$ and $G \in \mathcal{G}(\mathcal{I}_C(S))$.

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Thank you!