Gorenstein transpose and its dual

Guoqiang Zhao

January 11, 2019

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Introduction and motivation



The dual of the Gorenstein transpose

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1. Transpose (see [AB]) Let *R* be a left and right Northerian ring. For $M \in \text{mod } R$, there exists a projective presentation in mod *R*:

 $P_1 \xrightarrow{f} P_0 \to M \to 0.$

Then we get an exact sequence in mod *R*^{op}:

$$0 \to M^* \to P_0^* \xrightarrow{f^*} P_1^* \to \operatorname{coker} f^* \to 0,$$

where $()^* = \text{Hom}(-, R)$. coker f^* is called a *transpose* of M, and denoted by TrM.

Remark. The transpose of M depends on the choice of the projective presentation of M, but it is unique up to projective equivalence.

Gorenstein projective module

Auslander's original definition([AB]):

Let *R* be a left and right Noetherian ring and *M* a finitely generated *R*-module. Recall that *M* has *G*-dimension 0 (or *G*-dimM = 0) if $\text{Ext}^{i}(M, R) = 0 = \text{Ext}^{i}(M^{*}, R)$ for i > 0 and *M* is reflexive (i.e. $M \cong M^{**}$)

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A left *R*-module *M* ∈ Mod *R* is called *Gorenstein projective* [EJ] if there is an exact sequence

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

- of projective left *R*-modules with $M = \operatorname{coker}(P_1 \to P_0)$ and such that $\operatorname{Hom}(-, P)$ leaves the sequence exact for each projective left *R*-module *P*. Denote the class of all Gorenstein projective left *R*-modules by $\mathcal{GP}(R)$.
- A *f.g.* module *M* over a Noetherian ring is Gorenstein projective iff G-dimM = 0.

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- A *f.g.* module *M* over a Noetherian ring is Gorenstein projective iff G-dimM = 0.

• A left *R*-module *M* is called *Gorenstein injective* if there is an exact sequence

$$\cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

of injective left *R*-modules with $M = \ker(E^0 \rightarrow E^1)$ such that $\operatorname{Hom}(E, -)$ leaves the sequence exact whenever *E* is an injective left *R*-module.

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2. Gorenstein Transpose (see [HH]) Let *R* be a left and right Northerian ring. For $M \in \text{mod } R$, there exists a Gorenstein projective presentation in mod *R*:

 $G_1 \xrightarrow{g} G_0 \to M \to 0.$

Then we get an exact sequence in mod R^{op} :

$$0 \to M^* \to G_0^* \xrightarrow{g^*} G_1^* \to \operatorname{coker} g^* \to 0,$$

where $()^* = \text{Hom}(, R)$. coker g^* is called a *Gorenstein transpose* of *M*, and denoted by $\text{Tr}_G M$.

Question. The Gorenstein transpose of M depends on the choice of the Gorenstein projective presentation of M, is it unique up to Gorenstein projective equivalence?

Main results about the Gorenstein transpose 1. Establish a relation between a Gorenstein transpose of a module with a transpose of the same module.

[HH, Theorem 3.1]

Let $M \in \text{mod } R$ and $A \in \text{mod } R^{op}$. Then A is a Gorenstein transpose of M if and only if there exists an exact sequence $0 \to A \to \text{Tr}M \to G \to$ 0 in mod R^{op} with G Gorenstein projective.

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2. Provide a method to construct a Gorenstein transpose of a module from a transpose of the same module.

[HH, Corollary 3.2]

Let *G* be a Gorenstein projective module. Then $TrM \oplus G$ is a Gorenstein transpose of *M*.

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However, the following two questions remain unknown:

 Is it true that the Gorenstein transpose of a module is unique up to Gorenstein projective equivalence?

 Is it true that any Gorenstein transpose of a module can be obtained by directed sums of a transpose of the same module and a Gorenstein projective module?

To resolve the questions above, it maybe needs the new relations between the Gorenstein transpose and transpose.

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[ZS, Theorem 2.1]

Let *R* be a left and right Noetherian ring, $M \in \text{mod } R$. Then $\text{Tr}_G M$ is a transpose of *N*, where $N \in \text{Ext}(\mathcal{GP}(R), M)$ is an extension of a Gorenstein projective *R*-module by *M*, which means that there is an exact sequence $0 \to M \to N \to G \to 0$ in mod *R* with *G* Gorenstein projective.

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[ZS, Theorem 2.3]

Suppose that $M \in \text{mod } R$. Then, for any Gorenstein transpose of M, there exists an exact sequence $0 \rightarrow H \rightarrow \text{Tr}_G M \rightarrow \text{Tr} M \rightarrow 0$ in mod R^{op} with H Gorenstein projective.

Remark. We do not know whether the converse is true.

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[ZS, Corollary 2.4]

Any two Gorenstein transposes of $M \in \text{mod } R$ are Gorenstein projectively equivalent.

[ZS, Corollary 2.5]

If $M \in \text{mod } R$ has finite projective dimension, then, for any Gorenstein transpose of M, there is a Gorenstein projective modules G, such that $\text{Tr}_G M = \text{Tr} M \oplus G$.

Let *R* and *S* be associative rings with units. We use Mod *R* (resp. Mod. 979) to denote the class of left *m*-modules (resp. right smodules):

Definition ([HW])

An (R, S)-bimodule $C = {}_{R}C_{S}$ is called *semidualizing* if it satisfies the following.

(*a*) $_{R}C$ and C_{S} admit a resolution by finitely generated projective left *R*-modules and projective right *S*-modules, respectively.

(b) The maps $R \to \operatorname{Hom}_{S^{op}}(C, C)$ and $S \to \operatorname{Hom}_{R}(C, C)$ are isomorphisms.

(c) $\operatorname{Ext}_{R}^{i \ge 1}(C, C) = 0 = \operatorname{Ext}_{S^{op}}^{i \ge 1}(C, C).$

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The dual of the transpose

Definition ([TH])

Let $M \in \text{Mod } R$, and $0 \to M \to I^0 \stackrel{g}{\to} I^1$ be an injective resolution of M. We denote either $\text{Hom}_R(_RC_S, -)$ or $\text{Hom}_{S^{op}}(_RC_S, -)$ by ()*. So we get an exact sequence in Mod S:

$$0 \to M_* \to I^0_* \xrightarrow{g_*} I^1_* \to \operatorname{coker} g_* \to 0.$$

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The dual of the Gorenstein transpose

- How to dualize the Gorenstein transpose of modules appropriately? Replacing an injective resolution of *M* by a Gorenstein injective resolution of *M*? (Unfortunately)
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{cotranspose} $\stackrel{generalizing}{\longrightarrow}$ {a more general concept} \implies {to find an appropriate module instead of Gorenstein injective module}

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\mathcal{Y} -cotranspose

Let \mathcal{Y} be a subcategory of Mod R, and $U = {}_{R}U_{S}$ be a fixed (R, S)bimodule. For convenience, we denote either $\operatorname{Hom}_{R}({}_{R}U_{S},-)$ or $\operatorname{Hom}_{R}({}_{R}U_{S},-)$

Definition ([Z])

Suppose that *A* has an \mathcal{Y} -copresentation, that is, there exists an exact sequence $0 \to A \to Y^0 \xrightarrow{g} Y^1$ in Mod *R* with Y^0 , $Y^1 \in \mathcal{Y}$. Applying the functor ()_{*} to the sequence above induces an exact sequence in Mod *S*:

$$0 \to A_* \to Y^0_* \xrightarrow{g_*} Y^1_* \to \operatorname{coker} g_* \to 0.$$

We call $\operatorname{coker}_{g_*} a \mathcal{Y}$ -cotranspose of A with respect to U, and denoted by $\operatorname{cTr}_{\mathcal{Y}}^U A$.

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First relation

Let 1/2 be a generator or cogenerator for 32, we want to investigenerate religions between second memory and a contempose

[Z, Theorem 4.3]

Let $A \in \text{Mod } R$ and \mathcal{W} be a cogenerator for \mathcal{Y} . Assume that \mathcal{Y} is closed under extensions and $\text{Ext}^1_R(U, \mathcal{Y}) = 0$. (1) If M is a \mathcal{Y} -cotranspose of A with respect to U, then there is an exact sequence $0 \to M \to \operatorname{cTr}^U_{\mathcal{W}}A \to Y_* \to 0$ in Mod S with $\operatorname{cTr}^U_{\mathcal{W}}A$ a \mathcal{W} -cotranspose of A and $Y \in \mathcal{Y}$. (2) If \mathcal{Y} is U-coflexive, \mathcal{Y}_* is closed under kernel of epimorphism and $\operatorname{Tor}^S_1(U, \mathcal{Y}_*) = 0$, then the converse of (1) is true.

First relation

Let \mathcal{W} be a generator or cogenerator for \mathcal{Y} , we want to investigate the relations between \mathcal{Y} -cotranspose and \mathcal{W} -cotranspose.

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The definition of U-coreflexive

For a left *R*-module A, let $\theta_A : U \otimes_S A_* \to A$ via $\theta_A(x \otimes f) = f(x)$, for any $x \in U$ and $f \in A_*$, be the canonical evaluation homomorphism.

[TH, Definition 2.4]

A is called *U*-coreflexive if θ_A is an isomorphism.

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Motivated by [Z, Theorem 4.3], we introduce the following notion

Definition

A left *R*-module *M* is called \mathcal{L}_C -Gorenstein injective if there exists an exact sequence:

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

in $\mathcal{I}(R)$, such that $M \cong \operatorname{im}(I_0 \to I^0)$ and the sequence is $\operatorname{Hom}_R(\mathcal{I}(R), -)$ -exact and $\operatorname{Hom}_R(C, -)$ -exact. Denote the class of all \mathcal{L}_C -Gorenstein injective left *R*-modules by $\mathcal{L}_C(R)$.

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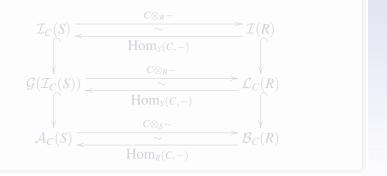
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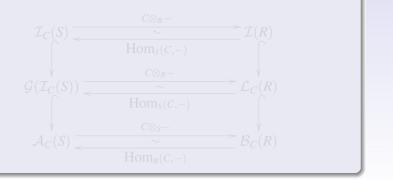
Fact

(1) $\mathcal{L}_C(R) = \mathcal{GI}(R) \cap \mathcal{B}_C(R)$. (2) There are Foxby equivalences of categories:



Fact

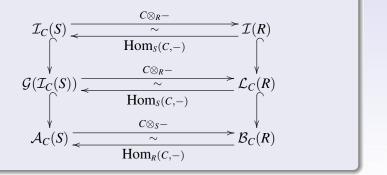
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Fact

(1) $\mathcal{L}_C(R) = \mathcal{GI}(R) \cap \mathcal{B}_C(R).$

(2) There are Foxby equivalences of categories:



Definition [HW]

The Bass class $\mathcal{B}_C(R)$ with respect to *C* is the subcategory of left *R*-modules *M* satisfying: (1) $\operatorname{Ext}_R^{i \ge 1}(C, M) = 0 = \operatorname{Tor}_{i \ge 1}^S(C, \operatorname{Hom}_R(C, M))$ and (2) The natural evaluation map $\theta_M : C \otimes_S \operatorname{Hom}_R(C, M) \to M$ is an isomorphism.

The Auslander class $\mathcal{A}_C(S)$ with respect to *C* is the subcategory of left *S*-modules *N* satisfying: (1) $\operatorname{Tor}_{i\geq 1}^{S}(C,N) = 0 = \operatorname{Ext}_{R}^{i\geq 1}(C,C\otimes_{S}N)$ and (2) The natural evaluation map $N \to \operatorname{Hom}_{R}(C,C\otimes_{S}N)$ is an isomorphism.

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Definition [HW]

The Bass class $\mathcal{B}_C(R)$ with respect to *C* is the subcategory of left *R*-modules *M* satisfying: (1) $\operatorname{Ext}_R^{i \ge 1}(C, M) = 0 = \operatorname{Tor}_{i \ge 1}^S(C, \operatorname{Hom}_R(C, M))$ and (2) The natural evaluation map $\theta_M : C \otimes_S \operatorname{Hom}_R(C, M) \to M$ is an isomorphism.

The Auslander class $\mathcal{A}_C(S)$ with respect to *C* is the subcategory of left *S*-modules *N* satisfying: (1) $\operatorname{Tor}_{i\geq 1}^S(C,N) = 0 = \operatorname{Ext}_R^{i\geq 1}(C,C\otimes_S N)$ and (2) The natural evaluation map $N \to \operatorname{Hom}_R(C,C\otimes_S N)$ is an isomorphism.

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Fact

(1) \$\mathcal{L}_C(R)\$ is closed under extension and injective resolving.
(2) \$\mathcal{I}(R)\$ are both a generator and cogenerator for \$\mathcal{L}_C(R)\$.
(3) For any \$i > 0\$, \$\mathbb{Ext}_R^i(C, \$\mathcal{L}_C(R)\$) = 0\$.

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(1) B_C(R) is closed under extension and injective resolving.
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To give the dual counterparts of Gorenstein transposes. Motivated by Theorem 4.3, we should choose the \mathcal{L}_C -Gorenstein injective copresentation instead of the Gorenstein minimum copresentation.

Definition

Let $A \in \text{Mod } R$. Then there exists an exact sequence $0 \to A \to G^0 \xrightarrow{g} G^1$ in Mod R with G^0 , $G^1 \in \mathcal{L}_C(R)$. Applying the functor $()_* = \text{Hom}_R(_RC_S, -)$ to the sequence above induces an exact sequence in Mod S:

$$0 \to A_* \to G^0_* \xrightarrow{g_*} G^1_* \to \operatorname{coker} g_* \to 0.$$

We call $coker_{g*}$ a \mathcal{L}_C -Gorenstein cotranspose of A with respect to C.

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The following result can be regarded as a dual of [HH, Theorem 3.11.

[Z, Theorem 4.5]

Let $A \in Mod R$. Then any \mathcal{L}_C -Gorenstein cotranspose of A can be embedded into a cotranspose of A with the cokernel in $\mathcal{G}(\mathcal{I}_C(S))$.

Remark. We do not know whether the converse is true. But we have

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Second relation

[Z, Theorem 4.11]

Let $A \in \text{Mod } R$. Assume that \mathcal{V} is a generator for \mathcal{Y} , and \mathcal{Y} is closed under extensions. If $\text{Ext}^1_R(U,\mathcal{Y}) = 0$, then, for any \mathcal{Y} -cotranspose $c\text{Tr}^U_{\mathcal{Y}}A$ of A, there is an isomorphism $c\text{Tr}^U_{\mathcal{Y}}A \cong c\text{Tr}^U_{\mathcal{V}}B$ for some $B \in \text{Ext}(A, \mathcal{Y})$.

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Any \mathcal{L}_C -Gorenstein cotranspose of A, is a cotranspose of B, where $B \in \text{Ext}(A, \mathcal{L}_C(R))$.

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[Z, Theorem 4.12]

Suppose that $A \in \text{Mod } R$. Then, for any \mathcal{L}_C -Gorenstein cotranspose of A, there exists an exact sequence $0 \to G \to c \text{Tr}_{\mathcal{L}_C} A \oplus E \to c \text{Tr} A \to 0$ in Mod S with $E \in \mathcal{I}_C(S)$ and $G \in \mathcal{G}(\mathcal{I}_C(S))$.

This can be regarded as a dual of [ZS, Theorem 2.3].

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By Theorem 4.11, we give another relation between a \mathcal{L}_C -Gorenstein cotranspose of a module and a cotranspose of the same module.

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Thank you!