Cohen-Macaulayness of invariant rings is determined by inertia groups

Ben Blum-Smith

Northeastern

November 30, 2018

Permutation invariants

$G \subset S_n$ acts on $\mathbb{Z}[x_1, \ldots, x_n]$. Problem: describe invariant subring.

Theorem (Fundamental Theorem on Symmetric Polynomials)

If $G = S_n$, then

$$\mathbb{Z}[x_1, \ldots, x_n]^G = \mathbb{Z}[\sigma_1, \ldots, \sigma_n],$$

where $\sigma_1 = \sum_i x_i$, $\sigma_2 = \sum_{i<j} x_i x_j$, etc.

What if $G \subsetneq S_n$?

Permutation invariants

Over $\mathbb{Q}$:

Theorem (Kronecker 1881)

$\mathbb{Q}[x_1, \ldots, x_n]^G$ is a free module over $\mathbb{Q}[\sigma_1, \ldots, \sigma_n]$.

Kronecker's contribution is not well-known, but a modern invariant theorist would see this as an immediate consequence of the Hochster-Eagon theorem.

Permutation invariants

Example

$G = \langle (1234) \rangle \subset S_4$, acting on $\mathbb{Q}[x, y, z, w]$.

$$
\begin{aligned}
g_0 &= 1 \\
g_2 &= xz + yw \\
g_3 &= x^2y + y^2z + \dots \\
g_{4a} &= x^2yz + y^2zw + \dots \\
g_{4b} &= xy^2z + yz^2w + \dots \\
g_5 &= x^2y^2z + y^2z^2w + \dots
\end{aligned}
$$

is a basis over $\mathbb{Q}[\sigma_1, \dots, \sigma_4]$.

$$x^3y^2z + y^3z^2w + \dots = \frac{1}{2}\sigma_3 g_3 - \frac{1}{2}\sigma_2 g_{4b} + \frac{1}{2}\sigma_1 g_5$$

Permutation invariants

Statement fails over $\mathbb{Z}$.

$$x^3y^2z + y^3z^2w + \cdots = \frac{1}{2}\sigma_3 g_3 - \frac{1}{2}\sigma_2 g_{4b} + \frac{1}{2}\sigma_1 g_5$$

Problem

For which $G \subset S_n$ does the statement of Kronecker's theorem hold over $\mathbb{Z}$?

Equivalent to:

Problem

For which $G \subset S_n$ is $k[x_1, \ldots, x_n]^G$ a Cohen-Macaulay ring for any field k?

Permutation invariants

Let $k[\mathbf{x}] := k[x_1, \ldots, x_n]$.

Theorem (BBS '17)

If $G \subset S_n$ is generated by transpositions, double transpositions, and 3-cycles, then

$$k[\mathbf{x}]^G$$

is Cohen-Macaulay regardless of k.

Theorem (BBS - Sophie Marques '18)

The converse is also true.

"If" direction sketch

- $k[\mathbf{x}]^G$ is CM if the invariants $k[\Delta]^G$ of an appropriate *Stanley-Reisner ring* $k[\Delta]$ are CM (Garsia-Stanton '84; Reiner '03), where Δ is a specific triangulation of a ball.
- $k[\Delta]^G \cong k[\Delta/G]$ (Reiner '90).
- CMness of $k[\Delta/G]$ is equivalent to a purely topological condition on Δ/G:
  - $\tilde{H}_i(\Delta/G; k) = 0$ for $i < n-1$, and
  - $H_i(\Delta/G, \Delta/G - p; k) = 0$ for $i < n-1$ and $p \in \Delta/G$.

  (Reisner '75; Munkres '84; Stanley '91; Duval '97)
- If G is generated by transpositions, double transpositions, and 3-cycles, then Δ/G is a ball (Lange '16).

Story of the "only-if" direction

By reasoning about the topological quotient Δ/G, I showed in my thesis (BBS '17) that if G is *not* generated by transpositions, double transpositions, and 3-cycles, then there is a field k such that $k[\Delta]^G$ is not CM.

However, the arguments of Garsia-Stanton '84 do not allow one to transfer this conclusion back to $k[\mathbf{x}]^G$.

After I defended, Sophie Marques proposed to transfer the *argument*, rather than the conclusion, from $k[\Delta]^G$ to $k[\mathbf{x}]^G$.

This necessitated a search for a commutative-algebraic fact to replace each topological fact we used.

Local structure in a quotient

Let X be a hausdorff topological space carrying an action by a finite group G. Let $x \in X$. Let G_x be the stabilizer of x for the action of G. Let X/G be the topological quotient, and let $\overline{x}$ be the image of x in X/G.

Theorem (local structure in a quotient)

There is a neighborhood U of x, invariant under G_x, such that U/G_x is homeomorphic to a neighborhood of $\overline{x}$ in X/G.

Proof: Pick U small enough so that if $gx \neq x$, then $gU \cap U = \emptyset$. Make it G_x-invariant by intersecting its G_x-images. Then the quotient map restricted to U factors through U/G_x and the induced map on U/G_x is injective. Since group quotient maps are open maps, this makes it a homeomorphism.

Local structure in a quotient

What is the commutative-algebraic analogue?

Let A be a ring.

- $x \in X$ becomes $\mathfrak{P} \triangleleft A$.
- X/G becomes A^G.
- $\overline{x} \in X/G$ becomes $\mathfrak{p} = \mathfrak{P} \cap A^G$.
- G_x becomes $I_G(\mathfrak{P}) := \{g \in G : a - ga \in \mathfrak{P},\ \forall a \in A\}$. (Not $D_G(\mathfrak{P})$!)
- The appropriate analogue for the sufficiently small neighborhood of $\overline{x}$ in X/G is the *strict henselization* of A^G at $\mathfrak{p}$.

Local structure in a quotient

Let C be a (commutative, unital) ring. Let $\mathfrak{p}$ be a prime ideal of C. The *strict henselization* of C at $\mathfrak{p}$ is a local ring $C_{\mathfrak{p}}^{hs}$ together with a local map $C_{\mathfrak{p}} \to C_{\mathfrak{p}}^{hs}$ with the following properties:

1. $C_{\mathfrak{p}}^{hs}$ is a henselian ring.
2. $\kappa(C_{\mathfrak{p}}^{hs})$ is the separable closure of $\kappa(C_{\mathfrak{p}})$.
3. $C_{\mathfrak{p}}$ and $C_{\mathfrak{p}}^{hs}$ are simultaneously noetherian (resp. CM).
4. $C_{\mathfrak{p}} \to C_{\mathfrak{p}}^{hs}$ is faithfully flat of relative dimension zero.

$C_{\mathfrak{p}}^{hs}$ is universal with respect to 1 and 2. It should be viewed as a "very small neighborhood of $\mathfrak{p}$ in C."

Local structure in a quotient

Let A be a ring with an action by a finite group G. Let $\mathfrak{p}$ be a prime of A^G. Let $C_\mathfrak{p}^{hs}$ be the strict henselization of A^G at $\mathfrak{p}$. Define

$$A_\mathfrak{p}^{hs} := A \otimes_{A^G} C_\mathfrak{p}^{hs}$$

Note G acts on $A_\mathfrak{p}^{hs}$ through its action on A.

Let $\mathfrak{P}$ be a prime of A lying over $\mathfrak{p}$ and let $\mathfrak{Q}$ be a prime of $A_\mathfrak{p}^{hs}$ pulling back to $\mathfrak{P}$. Recall $I_G(\mathfrak{P}) := \{g \in G : a - ga \in \mathfrak{P}, \forall a \in A\}$. (Fact: $I_G(\mathfrak{Q}) = I_G(\mathfrak{P})$.)

Theorem (Raynaud '70)

There is a ring isomorphism $(A_\mathfrak{p}^{hs})_\mathfrak{Q}^{I_G(\mathfrak{P})} \cong C_\mathfrak{p}^{hs}$.

This is the commutative-algebraic analogue!

Local structure in a quotient

Corollary (BBS - Marques '18)

Assume A^G is noetherian. Then TFAE:

1. *A^G is CM.*
2. *For every prime $\mathfrak{p}$ of A^G and every $\mathfrak{Q}$ of $A_{\mathfrak{p}}^{hs}$ pulling back to a $\mathfrak{P}$ of A lying over $\mathfrak{p}$,*
$$(A_{\mathfrak{p}}^{hs})_{\mathfrak{Q}}^{I_G(\mathfrak{P})}$$
is CM.
3. *For every maximal $\mathfrak{p}$ of A^G, there is some $\mathfrak{Q}$ of $A_{\mathfrak{p}}^{hs}$ pulling back to a $\mathfrak{P}$ of A lying over $\mathfrak{p}$, such that*
$$(A_{\mathfrak{p}}^{hs})_{\mathfrak{Q}}^{I_G(\mathfrak{P})}$$
is CM.

Permutation invariants - “only-if” direction

Back to the permutation group context. Let $k[\mathbf{x}] = \mathbb{F}_p[x_1, \dots, x_n]$, for some prime p to be determined later.

Let N be the subgroup of G generated by transpositions, double transpositions, and 3-cycles.

It suffices to find, when $N \subsetneq G$, a $\mathfrak{p} \lhd k[\mathbf{x}]^G$ such that the corresponding $C_{\mathfrak{p}}^{hs}$ is not CM.

Permutation invariants - "only-if" direction

Note that G/N acts on $k[\mathbf{x}]^N$.

Theorem (BBS - Marques '18)

If there is a prime $\mathfrak{P}$ of $k[\mathbf{x}]^N$ whose inertia group $I_{G/N}(\mathfrak{P})$ is a p-group, then $k[\mathbf{x}]^G$ is not CM.

(Recall $k = \mathbb{F}_p$.)

The main ingredients of the proof are:

- the above result which says that CMness at $\mathfrak{P} \cap k[\mathbf{x}]^G$ only depends on the action of $I_{G/N}(\mathfrak{P})$ on the appropriate strict henselization.
- a theorem of Lorenz and Pathak '01 which shows that such $I_{G/N}(\mathfrak{P})$ obstructs CMness.

It also uses the "if" direction to conclude that $k[\mathbf{x}]^N$ is CM. The invocation of Lorenz and Pathak needs this.

Permutation invariants - “only-if” direction

So the problem is reduced to finding a prime number p and a prime ideal $\mathfrak{P}$ of $k[\mathbf{x}]^N$ such that $I_{G/N}(\mathfrak{P})$ is a p-group, when $N \subsetneq G$.

Let Π_n be the poset of partitions of $[n]$, ordered by refinement.

Each $\pi \in \Pi_n$ corresponds to the ideal $\mathfrak{P}_\pi^\star$ of $k[\mathbf{x}]$ generated by $x_i - x_j$ for each pair i, j in the same block of π. (Cf. the braid arrangement.)

Let G_π^B be the blockwise stabilizer of π in G, and let $G_\pi^B N/N$ be its image in G/N. If $\mathfrak{P}_\pi = \mathfrak{P}_\pi^\star \cap k[\mathbf{x}]^N$, one can show that

$$I_{G/N}(\mathfrak{P}_\pi) = G_\pi^B N/N.$$

Permutation invariants - "only-if" direction

So we just need to find π such that $G_\pi^B N/N$ is a p-group.

Consider the map

$$\varphi : G \to \Pi_n$$

that sends a permutation g to the decomposition of $[n]$ into orbits of g.

Proposition (BBS '17)

If $g \in G \setminus N$ is such that $\pi = \varphi(g)$ is minimal in $\varphi(G \setminus N)$, then $G_\pi^B N/N$ has prime order (and is generated by the image of g).

If $N \subsetneq G$, $G \setminus N$ is nonempty, so such g exists, and fixing p as the order of $G_\pi^B N/N$, we find that $k[\mathbf{x}]^G$ is not CM. □

Thank you!