Quasi-abelian hearts of twin cotorsion pairs on triangulated categories

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Definition

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Notation:

$$\mathcal{U}*\mathcal{V} \coloneqq \{X \in \mathcal{C} \mid \exists \Delta \colon U \to X \to V \to \Sigma U, \text{ some } U \in \mathcal{U}, V \in \mathcal{V}\}$$

- $m{\circ}$ $\mathcal{C}=$ cluster category (triangulated, Hom-finite, Krull-Schmidt, has Serre duality)
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Aim: to study mod Λ_R

$Hom_{\mathcal{C}}(R, -)$ functor

$$\mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(R,-)} \mathsf{mod}\,\Lambda_R$$

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$Hom_{\mathcal{C}}(R, -)$ functor

$$\begin{array}{c}
\mathcal{C} \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(R,-)} \operatorname{mod} \Lambda_{R} \\
Q \downarrow & \operatorname{Hom}_{\mathcal{C}/\mathcal{X}_{R}}(R,-) \\
\mathcal{C}/\mathcal{X}_{R}
\end{array}$$

where

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Buan-Marsh-Reiten: Cluster-tilted algebras

Definition

 $T \in \mathcal{C}$ is called a *cluster-tilting* object if T is rigid and has a maximal number of non-isomorphic direct summands.

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Let $T \in \mathcal{C}$ be a cluster-tilting object. Then there is an equivalence of categories

$$\mathcal{C}/\mathcal{X}_T \simeq \mathsf{mod}\,\Lambda_T$$
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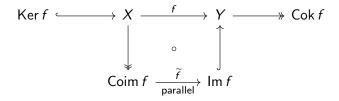
Note that for a cluster-tilting object, we have $\mathcal{X}_T = \operatorname{add} \Sigma T$.

Preabelian categories

• A category is *preabelian* if it admits kernels and cokernels.

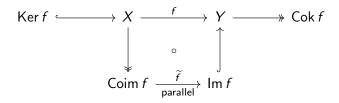
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• Such a category is semi-abelian if \widetilde{f} is regular, i.e. simultaneously monic and epic, for all f.

Buan-Marsh: calculus of fractions

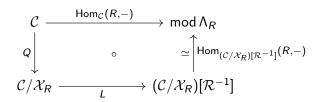
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Note:
$$S = {}^{\perp_1}\mathcal{T} \coloneqq \{X \in \mathcal{C} \mid \operatorname{Ext}^1_{\mathcal{C}}(X, \mathcal{T}) = 0\}$$
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Example

The pair $(\operatorname{add} \Sigma R, \mathcal{X}_R)$ is a cotorsion pair.

Recall
$$\mathcal{X}_R = \{X \in \mathcal{C} \mid \mathsf{Hom}_{\mathcal{C}}(R, X) = 0\}.$$



Let $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}$ be nice subcategories of \mathcal{C} .

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Example

If (S, T) is a cotorsion pair, then ((S, T), (S, T)) is a *degenerate* twin cotorsion pair.

Nakaoka: the heart

Assume from now that $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a twin cotorsion pair, and define

$$\mathcal{W} \coloneqq \mathcal{T} \cap \mathcal{U},$$

$$\mathcal{C}^- \coloneqq \Sigma^{-1} \mathcal{S} * \mathcal{W},$$

$$\mathcal{C}^+ := \mathcal{W} * \Sigma \mathcal{V}.$$

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The associated *heart* is defined to be

$$\overline{\mathcal{H}} \coloneqq \mathcal{C}^- \cap \mathcal{C}^+/\mathcal{W}$$

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$$\begin{split} \mathcal{W} &\coloneqq \mathcal{T} \cap \mathcal{U}, \\ \mathcal{C}^- &\coloneqq \Sigma^{-1} \mathcal{S} * \mathcal{W}, \\ \mathcal{C}^+ &\coloneqq \mathcal{W} * \Sigma \mathcal{V}. \end{split}$$

The associated *heart* is defined to be

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and is semi-abelian.

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• If
$$((S,T),(U,V))=((\operatorname{add}\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,\mathcal{X}_R^{\perp_1}))$$
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• If $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U} * \mathcal{V}$, then $\overline{\mathcal{H}}$ is integral.

Main Result

A *quasi-abelian* category is a semi-abelian category in which PBs of cokernels are cokernels and POs of kernels are kernels.

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Theorem (S.)

Let $((S, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ be a twin cotorsion pair on a triangulated category \mathcal{C} . If $\mathcal{U} \subseteq \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U}$, then $\overline{\mathcal{H}}$ is quasi-abelian.

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A *quasi-abelian* category is a semi-abelian category in which PBs of cokernels are cokernels and POs of kernels are kernels.

Theorem (S.)

Let ((S, T), (U, V)) be a twin cotorsion pair on a triangulated category C. If $U \subseteq T$ or $T \subseteq U$, then $\overline{\mathcal{H}}$ is quasi-abelian.

Setting $((S, T), (U, V)) = ((\text{add } \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, \mathcal{X}_R^{\perp_1}))$, we get

Corollary

 $\mathcal{C}/\mathcal{X}_R$ is quasi-abelian.



An interesting consequence!

• $\mathcal{C}/\mathcal{X}_R$ is quasi-abelian: a bunch of Auslander-Reiten theory applies! [S.]

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 \bullet $\mathcal{C}/\mathcal{X}_R$ is Krull-Schmidt: a bunch of Auslander-Reiten theory applies! [Liu]

E.g. the AR theory of C induces the AR theory of C/\mathcal{X}_R .

Quasi-abelian meets Krull-Schmidt

Theorem (S.)

Let \mathcal{A} be a Krull-Schmidt quasi-abelian category, and $\xi\colon X\stackrel{f}{\to}Y\stackrel{g}{\to}Z$ an exact sequence in \mathcal{A} . The following are equivalent:

- (i) ξ is an Auslander-Reiten sequence;
- (ii) $End_A(X)$ is local and g is right almost split;
- (iii) End_A(Z) is local and f is left almost split;
- (iv) f is minimal left almost split; and
- (v) g is minimal right almost split.