Quasi-abelian hearts of twin cotorsion pairs on triangulated categories

Amit Shah

University of Leeds

Northeastern University
01 May 2018

Triangulated categories

Definition

A *triangulated* category is an additive category $\mathcal{C}$, together with a collection of *triangles*

$$X \to Y \to Z \to \Sigma X,$$

where Σ is an automorphism of $\mathcal{C}$ and the triangles satisfy some axioms.

Triangulated categories

Definition

A *triangulated* category is an additive category $\mathcal{C}$, together with a collection of *triangles*

$$X \to Y \to Z \to \Sigma X,$$

where Σ is an automorphism of $\mathcal{C}$ and the triangles satisfy some axioms.

Σ is called the *shift* or *suspension* functor.

Triangulated categories

Definition

A *triangulated* category is an additive category $\mathcal{C}$, together with a collection of *triangles*

$$X \to Y \to Z \to \Sigma X,$$

where Σ is an automorphism of $\mathcal{C}$ and the triangles satisfy some axioms.

Σ is called the *shift* or *suspension* functor.

We'll only need the following axiom: any morphism $f : X \to Y$ may be completed to a triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$.

Triangulated categories

Definition

A *triangulated* category is an additive category $\mathcal{C}$, together with a collection of *triangles*

$$X \to Y \to Z \to \Sigma X,$$

where Σ is an automorphism of $\mathcal{C}$ and the triangles satisfy some axioms.

Σ is called the *shift* or *suspension* functor.

We'll only need the following axiom: any morphism $f\colon X \to Y$ may be completed to a triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$.

Notation:

$$\mathcal{U} * \mathcal{V} := \{X \in \mathcal{C} \mid \exists \Delta\colon U \to X \to V \to \Sigma U, \text{ some } U \in \mathcal{U}, V \in \mathcal{V}\}$$

Set-up & Motivation

- $\mathcal{C}$ = cluster category (triangulated, Hom-finite, Krull-Schmidt, has Serre duality)
- Σ = shift functor

Set-up & Motivation

- $\mathcal{C}$ = cluster category (triangulated, Hom-finite, Krull-Schmidt, has Serre duality)
- Σ = shift functor
- R = *rigid*, i.e. $\mathrm{Ext}^1_{\mathcal{C}}(R, R) := \mathrm{Hom}_{\mathcal{C}}(R, \Sigma R) = 0$, object of $\mathcal{C}$

Set-up & Motivation

- $\mathcal{C}$ = cluster category (triangulated, Hom-finite, Krull-Schmidt, has Serre duality)

- Σ = shift functor

- R = *rigid*, i.e. $\mathrm{Ext}^1_{\mathcal{C}}(R, R) := \mathrm{Hom}_{\mathcal{C}}(R, \Sigma R) = 0$, object of $\mathcal{C}$

- $\Lambda_R := (\mathrm{End}_{\mathcal{C}}\, R)^{\mathrm{op}}$

Set-up & Motivation

- $\mathcal{C}$ = cluster category (triangulated, Hom-finite, Krull-Schmidt, has Serre duality)
- Σ = shift functor
- R = *rigid*, i.e. $\operatorname{Ext}^1_{\mathcal{C}}(R, R) := \operatorname{Hom}_{\mathcal{C}}(R, \Sigma R) = 0$, object of $\mathcal{C}$
- $\Lambda_R := (\operatorname{End}_{\mathcal{C}} R)^{\mathrm{op}}$

Aim: to study $\operatorname{mod} \Lambda_R$

$\mathrm{Hom}_{\mathcal{C}}(R,-)$ functor

$$\mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(R,-)} \mathrm{mod}\,\Lambda_R$$

$$\mathcal{X}_R = \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(R,X) = 0\}$$

$\mathrm{Hom}_{\mathcal{C}}(R,-)$ functor

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(R,-)} & \operatorname{mod}\Lambda_R \\
{\scriptstyle Q}\downarrow & \nearrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}/\mathcal{X}_R}(R,-)} & \\
\mathcal{C}/\mathcal{X}_R & &
\end{array}$$

where

$$\mathcal{X}_R = \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(R,X) = 0\}$$

Buan-Marsh-Reiten: Cluster-tilted algebras

Definition

$T \in \mathcal{C}$ is called a *cluster-tilting* object if T is rigid and has a maximal number of non-isomorphic direct summands.

Buan-Marsh-Reiten: Cluster-tilted algebras

Definition

$T \in \mathcal{C}$ is called a *cluster-tilting* object if T is rigid and has a maximal number of non-isomorphic direct summands.

Theorem

Let $T \in \mathcal{C}$ be a cluster-tilting object. Then there is an equivalence of categories

$$\mathcal{C}/\mathcal{X}_T \simeq \mathit{mod}\,\Lambda_T.$$

Buan-Marsh-Reiten: Cluster-tilted algebras

Definition

$T \in \mathcal{C}$ is called a *cluster-tilting* object if T is rigid and has a maximal number of non-isomorphic direct summands.

Theorem

Let $T \in \mathcal{C}$ be a cluster-tilting object. Then there is an equivalence of categories

$$\mathcal{C}/\mathcal{X}_T \simeq \mathit{mod}\, \Lambda_T.$$

Note that for a cluster-tilting object, we have $\mathcal{X}_T = \operatorname{add} \Sigma T$.

Preabelian categories

- A category is *preabelian* if it admits kernels and cokernels.

Preabelian categories

- A category is *preabelian* if it admits kernels and cokernels.

$$\begin{array}{ccccccc} \operatorname{Ker} f & \hookrightarrow & X & \xrightarrow{\ f\ } & Y & \twoheadrightarrow & \operatorname{Cok} f \\ & & \downarrow & \circ & \uparrow & & \\ & & \operatorname{Coim} f & \xrightarrow[\text{parallel}]{\ \widetilde{f}\ } & \operatorname{Im} f & & \end{array}$$

Preabelian categories

- A category is *preabelian* if it admits kernels and cokernels.

$$
\begin{array}{ccccccc}
\operatorname{Ker} f & \hookrightarrow & X & \xrightarrow{f} & Y & \twoheadrightarrow & \operatorname{Cok} f \\
 & & \downarrow\!\!\downarrow & \circ & \uparrow & & \\
 & & \operatorname{Coim} f & \xrightarrow[\text{parallel}]{\widetilde{f}} & \operatorname{Im} f & &
\end{array}
$$

- Such a category is *semi-abelian* if $\widetilde{f}$ is *regular*, i.e. simultaneously monic and epic, for all f.

Buan-Marsh: calculus of fractions

- $\mathcal{C}/\mathcal{X}_R$ is an *integral* category, i.e. semi-abelian, and PBs of epics are epics and POs of monics are monic.

Buan-Marsh: calculus of fractions

- $\mathcal{C}/\mathcal{X}_R$ is an *integral* category, i.e. semi-abelian, and PBs of epics are epics and POs of monics are monic.

- The class $\mathcal{R}$ of all regular morphisms admits a calculus of fractions. [Rump]

Buan-Marsh: calculus of fractions

- $\mathcal{C}/\mathcal{X}_R$ is an *integral* category, i.e. semi-abelian, and PBs of epics are epics and POs of monics are monic.

- The class $\mathcal{R}$ of all regular morphisms admits a calculus of fractions. [Rump]

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(R,-)} & \operatorname{mod}\Lambda_R \\
{\scriptstyle Q}\big\downarrow & \circ & \big\uparrow{\scriptstyle \simeq\ \operatorname{Hom}_{(\mathcal{C}/\mathcal{X}_R)[\mathcal{R}^{-1}]}(R,-)} \\
\mathcal{C}/\mathcal{X}_R & \xrightarrow[L]{} & (\mathcal{C}/\mathcal{X}_R)[\mathcal{R}^{-1}]
\end{array}$$

Nakaoka: Cotorsion pairs

Let $\mathcal{S}, \mathcal{T}$ be nice subcategories of $\mathcal{C}$.

Definition

A *cotorsion pair* is a pair $(\mathcal{S}, \mathcal{T})$ such that:

Nakaoka: Cotorsion pairs

Let $\mathcal{S}, \mathcal{T}$ be nice subcategories of $\mathcal{C}$.

Definition

A *cotorsion pair* is a pair $(\mathcal{S}, \mathcal{T})$ such that:

- $\mathcal{C} = \mathcal{S} * \Sigma\mathcal{T} := \{X \in \mathcal{C} \mid \exists \Delta : \ S \to X \to \Sigma T \to \Sigma S, \ S \in \mathcal{S}, T \in \mathcal{T}\}$

Nakaoka: Cotorsion pairs

Let $\mathcal{S}, \mathcal{T}$ be nice subcategories of $\mathcal{C}$.

Definition

A *cotorsion pair* is a pair $(\mathcal{S}, \mathcal{T})$ such that:

- $\mathcal{C} = \mathcal{S} * \Sigma\mathcal{T} := \{X \in \mathcal{C} \mid \exists\Delta : \; S \to X \to \Sigma T \to \Sigma S, \; S \in \mathcal{S}, T \in \mathcal{T}\}$
- $\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{S}, \mathcal{T}) = 0$

Nakaoka: Cotorsion pairs

Let $\mathcal{S}, \mathcal{T}$ be nice subcategories of $\mathcal{C}$.

Definition

A *cotorsion pair* is a pair $(\mathcal{S}, \mathcal{T})$ such that:

- $\mathcal{C} = \mathcal{S} * \Sigma\mathcal{T} := \{X \in \mathcal{C} \mid \exists\Delta : \; S \to X \to \Sigma T \to \Sigma S, \; S \in \mathcal{S}, \; T \in \mathcal{T}\}$
- $\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{S}, \mathcal{T}) = 0$

Note: $\mathcal{S} = {}^{\perp_1}\mathcal{T} := \{X \in \mathcal{C} \mid \operatorname{Ext}^1_{\mathcal{C}}(X, \mathcal{T}) = 0\}$, and $\mathcal{T} = \mathcal{S}^{\perp_1}$.

Nakaoka: Cotorsion pairs

Let $\mathcal{S}, \mathcal{T}$ be nice subcategories of $\mathcal{C}$.

Definition

A *cotorsion pair* is a pair $(\mathcal{S}, \mathcal{T})$ such that:

- $\mathcal{C} = \mathcal{S} * \Sigma\mathcal{T} := \{X \in \mathcal{C} \mid \exists\Delta : \ S \to X \to \Sigma T \to \Sigma S, \ S \in \mathcal{S}, T \in \mathcal{T}\}$
- $\operatorname{Ext}^1_{\mathcal{C}}(\mathcal{S}, \mathcal{T}) = 0$

Note: $\mathcal{S} = {}^{\perp_1}\mathcal{T} := \{X \in \mathcal{C} \mid \operatorname{Ext}^1_{\mathcal{C}}(X, \mathcal{T}) = 0\}$, and $\mathcal{T} = \mathcal{S}^{\perp_1}$.

Example

The pair $(\operatorname{add}\Sigma R, \mathcal{X}_R)$ is a cotorsion pair.

Recall $\mathcal{X}_R = \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(R, X) = 0\}$.

Nakaoka: Twin cotorsion pairs

Let $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}$ be nice subcategories of $\mathcal{C}$.

Definition

A *twin cotorsion pair* is a pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ such that:

- $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ are each cotorsion pairs

Nakaoka: Twin cotorsion pairs

Let $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}$ be nice subcategories of $\mathcal{C}$.

Definition

A *twin cotorsion pair* is a pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ such that:

- $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ are each cotorsion pairs
- $\mathcal{S} \subseteq \mathcal{U}$

Nakaoka: Twin cotorsion pairs

Let $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}$ be nice subcategories of $\mathcal{C}$.

Definition

A *twin cotorsion pair* is a pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ such that:

- $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ are each cotorsion pairs
- $\mathcal{S} \subseteq \mathcal{U}$

Example

$$((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) = ((\operatorname{add} \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, \mathcal{X}_R^{\perp_1}))$$

Nakaoka: Twin cotorsion pairs

Let $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}$ be nice subcategories of $\mathcal{C}$.

Definition

A *twin cotorsion pair* is a pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ such that:

- $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ are each cotorsion pairs
- $\mathcal{S} \subseteq \mathcal{U}$

Example

$$((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) = ((\operatorname{add} \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, \mathcal{X}_R^{\perp_1}))$$

Example

If $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair, then $((\mathcal{S}, \mathcal{T}), (\mathcal{S}, \mathcal{T}))$ is a *degenerate* twin cotorsion pair.

Nakaoka: the heart

Assume from now that $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a twin cotorsion pair, and define

$$\mathcal{W} := \mathcal{T} \cap \mathcal{U},$$

$$\mathcal{C}^- := \Sigma^{-1}\mathcal{S} * \mathcal{W},$$

$$\mathcal{C}^+ := \mathcal{W} * \Sigma\mathcal{V}.$$

Nakaoka: the heart

Assume from now that $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a twin cotorsion pair, and define

$$\mathcal{W} := \mathcal{T} \cap \mathcal{U},$$

$$\mathcal{C}^- := \Sigma^{-1}\mathcal{S} * \mathcal{W},$$

$$\mathcal{C}^+ := \mathcal{W} * \Sigma\mathcal{V}.$$

The associated *heart* is defined to be

$$\overline{\mathcal{H}} := \mathcal{C}^- \cap \mathcal{C}^+/\mathcal{W}$$

Nakaoka: the heart

Assume from now that $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ is a twin cotorsion pair, and define

$$\mathcal{W} := \mathcal{T} \cap \mathcal{U},$$

$$\mathcal{C}^{-} := \Sigma^{-1}\mathcal{S} * \mathcal{W},$$

$$\mathcal{C}^{+} := \mathcal{W} * \Sigma\mathcal{V}.$$

The associated *heart* is defined to be

$$\overline{\mathcal{H}} := \mathcal{C}^{-} \cap \mathcal{C}^{+}/\mathcal{W}$$

and is semi-abelian.

Nakaoka: recovering Buan-Marsh

Recall: Buan-Marsh show $\mathcal{C}/\mathcal{X}_R$ is integral.

Nakaoka: recovering Buan-Marsh

Recall: Buan-Marsh show $\mathcal{C}/\mathcal{X}_R$ is integral.

- If $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\operatorname{add}\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,\mathcal{X}_R^{\perp_1}))$,

Nakaoka: recovering Buan-Marsh

Recall: Buan-Marsh show $\mathcal{C}/\mathcal{X}_R$ is integral.

- If $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\operatorname{add}\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,\mathcal{X}_R^{\perp_1}))$, then

$$\overline{\mathcal{H}} = \mathcal{C}/\mathcal{X}_R.$$

Nakaoka: recovering Buan-Marsh

Recall: Buan-Marsh show $\mathcal{C}/\mathcal{X}_R$ is integral.

- If $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\operatorname{add}\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,\mathcal{X}_R^{\perp_1}))$, then
$$\overline{\mathcal{H}} = \mathcal{C}/\mathcal{X}_R.$$

- If $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U} * \mathcal{V}$, then $\overline{\mathcal{H}}$ is integral.

Main Result

A *quasi-abelian* category is a semi-abelian category in which PBs of cokernels are cokernels and POs of kernels are kernels.

Main Result

A *quasi-abelian* category is a semi-abelian category in which PBs of cokernels are cokernels and POs of kernels are kernels.

Theorem (S.)

Let $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ *be a twin cotorsion pair on a triangulated category* $\mathcal{C}$*. If* $\mathcal{U}\subseteq\mathcal{T}$ *or* $\mathcal{T}\subseteq\mathcal{U}$*, then* $\overline{\mathcal{H}}$ *is quasi-abelian.*

Main Result

A *quasi-abelian* category is a semi-abelian category in which PBs of cokernels are cokernels and POs of kernels are kernels.

Theorem (S.)

Let $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ be a twin cotorsion pair on a triangulated category $\mathcal{C}$. If $\mathcal{U} \subseteq \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U}$, then $\overline{\mathcal{H}}$ is quasi-abelian.

Setting $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\operatorname{add}\Sigma R, \mathcal{X}_R),(\mathcal{X}_R, \mathcal{X}_R^{\perp_1}))$, we get

Corollary

$\mathcal{C}/\mathcal{X}_R$ is quasi-abelian.

An interesting consequence!

- $\mathcal{C}/\mathcal{X}_R$ is quasi-abelian: a bunch of Auslander-Reiten theory applies! [S.]

E.g. irreducible maps must be proper epic or proper monic.

An interesting consequence!

- $\mathcal{C}/\mathcal{X}_R$ is quasi-abelian: a bunch of Auslander-Reiten theory applies! [S.]

E.g. irreducible maps must be proper epic or proper monic.

- $\mathcal{C}/\mathcal{X}_R$ is Krull-Schmidt: a bunch of Auslander-Reiten theory applies! [Liu]

E.g. the AR theory of $\mathcal{C}$ induces the AR theory of $\mathcal{C}/\mathcal{X}_R$.

Quasi-abelian meets Krull-Schmidt

Theorem (S.)

Let $\mathcal{A}$ be a Krull-Schmidt quasi-abelian category, and $\xi\colon\ X \xrightarrow{f} Y \xrightarrow{g} Z$ an exact sequence in $\mathcal{A}$. The following are equivalent:

(i) *ξ is an Auslander-Reiten sequence;*

(ii) *$\mathrm{End}_{\mathcal{A}}(X)$ is local and g is right almost split;*

(iii) *$\mathrm{End}_{\mathcal{A}}(Z)$ is local and f is left almost split;*

(iv) *f is minimal left almost split; and*

(v) *g is minimal right almost split.*