AC-Gorenstein homological algebra and AC-Gorenstein rings

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References

This talk is mainly based on the papers:

- Daniel Bravo, James Gillespie, and Mark Hovey *The stable module category of a general ring*, arXiv:1405.5768.
- James Gillespie, *AC-Gorenstein rings and their stable module categories*, arXiv:1710.09899.

Outline

1. **Absolutely clean (AC) and level modules - Duality**
2. **Injective (resp. projective) abelian model structures**
3. **Gorenstein AC-injective (resp. AC-projective) modules**
4. **AC-Gorenstein rings**

Modules of type FP_∞

Main Idea: Different notions of "finite" module leads to different notions of injectivity and flatness.

Definition: An R-module M is said to be of type FP_∞ if it has a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is finitely generated.

Note: M of type $FP_\infty \implies M$ f.p. $\implies M$ f.g. But...

FP_∞ for Noetherian rings

Let R be a ring. The following are equivalent:

1. R is (left) Noetherian.
2. All finitely generated (left) modules are of type FP_∞.
3. The class of finitely generated (left) modules is thick. That is, it is closed under retracts and satisfies the 2 out of 3 property for short exact sequences.

Injective modules over Noetherian rings

This is the REASON behind the following well-known Fact:

Fact: The following are equivalent.

1. R is (left) Noetherian.
2. The class of injective (left) R-modules is closed under direct limits (or just direct sums).

(**Proof** of 1. implies 2.) For any ring, Baer's criterion implies that I is injective iff $\mathrm{Ext}^1_R(M, I) = 0$ for any finitely generated M. If R is Noetherian and M is finitely generated, take a projective resolution $P_* \to M$ with each P_n finitely generated (so f.p). Then...

$$\mathrm{Ext}^1_R(M, \varinjlim I_\alpha) = H^1[\mathrm{Hom}_R(P_*, \varinjlim I_\alpha)] \cong H^1[\varinjlim \mathrm{Hom}_R(P_*, I_\alpha)]$$

$$\cong \varinjlim H^1 \mathrm{Hom}_R(P_*, I_\alpha) \cong \varinjlim \mathrm{Ext}^1_R(M, I_\alpha) = 0$$

FP_∞ for coherent rings

Let R be a ring. The following are equivalent:

1. R is (left) coherent.
2. All finitely presented (left) modules are of type FP_∞.
3. The class of finitely presented (left) modules is thick.

This is the REASON behind the following Fact:

Fact: The following are equivalent.

1. R is (left) coherent.
2. The class of absolutely pure (left) R-modules is closed under direct limits.

(**Reason** 1. implies 2.) Absolutely pure means $\mathrm{Ext}^1_R(M, A) = 0$ for all f.p. M. So...

$$\mathrm{Ext}^1_R(M, \varinjlim A_\alpha) = H^1[\mathrm{Hom}_R(P_*, \varinjlim A_\alpha)] \cong H^1[\varinjlim \mathrm{Hom}_R(P_*, A_\alpha)]$$

$$\cong \varinjlim H^1 \mathrm{Hom}_R(P_*, A_\alpha) \cong \varinjlim \mathrm{Ext}^1_R(M, A_\alpha) = 0$$

Absolutely clean modules

Fact: Bieri (Geometric Group Theory) showed that the class of FP_∞-modules is thick over ANY ring R. This suggests...

Definition: An R-module N is called **absolutely clean** (or FP_∞-injective) if $\text{Ext}^1_R(M, N) = 0$ for every M of type FP_∞.

We call a s.e.s. $0 \to N \to A \to B \to 0$ "clean" if it remains exact after applying $\text{Hom}_R(M, -)$ for any M of type FP_∞. So...

Fact: A module N is absolutely clean if and only if every short exact sequence $0 \to N \to A \to B \to 0$ is clean.

Absolutely clean modules over coherent and Noetherian rings

Recall: M of type $FP_\infty \implies M$ f.p. $\implies M$ f.g.

So: I injective $\implies I$ absolutely pure $\implies I$ absolutely clean.

But...

Fact: (1) R is (left) coherent $\Leftrightarrow$ The absolutely clean (left) modules coincide with the absolutely pure modules.

(2) R is (left) Noetherian $\Leftrightarrow$ The absolutely clean (left) modules coincide with the injective modules.

Properties of absolutely clean modules

Proposition: Absolutely clean modules have the following properties over ANY ring R.

- The class of absolutely clean modules is closed under pure submodules and pure quotients.
- The class of absolutely clean modules is coresolving; that is, it contains the injectives and is closed under extensions and cokernels of monomorphisms.
- The class of absolutely clean modules is closed under products, **sums**, retracts, **direct limits**, and transfinite extensions.
- There is some regular cardinal κ such that every absolutely clean module is a transfinite extension of absolutely clean modules with cardinality bounded by κ.

Flat modules over coherent rings

All of what we have done has a "dual".

Recall...

Chase's Theorem: The following are equivalent.

1. R is (right) coherent.
2. The class of flat (left) R-modules is closed under products.

(**Reason** 1. implies 2.) Because... For any ring R we have F is flat iff $\mathrm{Tor}_1^R(M, F) = 0$ for all finitely presented M. So, if R is coherent,

$$\mathrm{Tor}_1^R(M, \prod F_\alpha) = H_1(P_* \otimes_R \prod F_\alpha) \cong H_1(\prod (P_* \otimes_R F_\alpha))$$

$$\cong \prod H_1(P_* \otimes_R F_\alpha) = \prod \mathrm{Tor}_1^R(M, F_\alpha) = 0$$

Level modules

This suggests the following generalization of flat modules:

Definition: A (left) R-module N is called **level** if $\text{Tor}_1^R(M, N) = 0$ for every (right) R-module M of type FP_∞.

Note: N flat $\implies$ N level.

But...

Prop: R is (right) coherent $\Leftrightarrow$ the level (left) modules coincide with the flat modules.

So... Level $=$ Flat for (right) Noetherian rings too.

Proposition: Level modules have the following properties over ANY ring R.

- The class of level modules is closed under pure submodules and pure quotients.
- The class of level modules is resolving; that is, it contains the projectives and is closed under extensions and kernels of epimorphisms.
- The class of level modules is closed under **products**, sums, retracts, direct limits, and transfinite extensions.
- There is some regular cardinal κ such that every level module is a transfinite extension of level modules with cardinality bounded by κ.

Properties of level modules

Theorem: Let R be ANY ring.

- (Level, Absolutely clean) form a "complete duality pair" with respect to taking character modules $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

  $$L \text{ level} \leftrightarrow L^+ \text{ absolutely clean}$$

  $$A \text{ absolutely clean} \leftrightarrow A^+ \text{ level}$$

- Every module has a level cover and the level modules form the left half of a complete hereditary cotorsion pair.

Outline

1. **Absolutely clean (AC) and level modules - Duality**
2. **Injective (resp. projective) abelian model structures**
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Cotorsion Pairs

Definition of Complete Cotorsion Pair.

Let $\mathcal{A}$ be an abelian category, such as R-Mod or $\mathrm{Ch}(R)$.

- A pair of classes $(\mathcal{X}, \mathcal{Y})$ of objects in $\mathcal{A}$ is a **cotorsion pair** if the following conditions hold:
  - $X \in \mathcal{X}$ iff $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0$ for all $Y \in \mathcal{Y}$.
  - $Y \in \mathcal{Y}$ iff $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0$ for all $X \in \mathcal{X}$.
- We say the cotorsion pair is **complete** if for any $A \in \mathcal{A}$ there exist short exact sequences $Y \rightarrowtail X \twoheadrightarrow A$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and $A \rightarrowtail Y' \twoheadrightarrow X'$ with $X' \in \mathcal{X}$ and $Y' \in \mathcal{Y}$.
- We say the cotorsion pair is **hereditary** if each of the following hold:
  1. For any s.e.s. $A \rightarrowtail X' \twoheadrightarrow X$, if $X, X' \in \mathcal{X}$ then also $A \in \mathcal{X}$.
  2. For any s.e.s. $Y \rightarrowtail Y' \twoheadrightarrow B$, if $Y, Y' \in \mathcal{Y}$ then also $B \in \mathcal{Y}$.

Connection to abelian model categories

IDEA: The theory of abelian model categories provides a powerful method for constructing triangulated categories. Through Hovey's correspondence, there is a bijective correspondence between "injective cotorsion pairs" on $\mathcal{A}$ and "injective abelian model structures" on $\mathcal{A}$.

Definition: If $\mathcal{A}$ has enough injectives, then we call a cotorsion pair $(\mathcal{W}, \mathcal{F})$ an **injective cotorsion pair** if it is complete, $\mathcal{W}$ is *thick*, and $\mathcal{W} \cap \mathcal{F}$ coincides with the class of injective objects.

The objects in $\mathcal{F}$ are then called **fibrant** objects.

If $\mathcal{A}$ has enough projectives, there is also the dual notion of a **projective cotorsion pair** $(\mathcal{C}, \mathcal{V})$ with **cofibrant** objects $\mathcal{C}$.

Fundamental Facts on (Injective) Model Structures

What is the point?

FACTS: Assume $\mathcal{M} = (\mathcal{W}, \mathcal{F})$ is an injective cotorsion pair on an abelian category $\mathcal{A}$. The following fundamental facts hold:

- There is an associated **homotopy category**, $\text{Ho}(\mathcal{M})$.
  1. Objects in $\text{Ho}(\mathcal{M})$ are the same as the objects in $\mathcal{A}$.
  2. $\text{Hom}_{\text{Ho}(\mathcal{M})}(X, Y) = \text{Hom}_{\mathcal{A}}(FX, FY)/\sim$ where $f \sim g$ iff $g - f$ factors through an injective object.
- There is a canonical functor $\gamma\colon \mathcal{A} \to \text{Ho}(\mathcal{M})$ which is a "triangulated localization" with respect to the trivial objects $\mathcal{W}$. That is, $\text{Ho}(\mathcal{M})$ is triangulated, the functor sends short exact sequences to exact triangles, and γ is universally initial among such functors taking $\mathcal{W}$ to 0.
- The full subcategory $\mathcal{F}$ is a Frobenius category and there is a triangulated equivalence of categories: $\text{Ho}(\mathcal{M}) \cong \mathcal{F}/\sim$.

Examples of Injective Cotorsion Pairs in Ch(R)

The injective model structure

$(\mathcal{E}, dg\widetilde{\mathcal{I}}) =$ (Exact complexes, DG-Injective complexes) in Ch(R).

$$\text{Ho}[\text{Ch}(R)] = \mathcal{D}(R) \simeq dg\widetilde{\mathcal{I}}/\sim$$

The Inj model structure

$(\mathcal{W}_1, dw\widetilde{\mathcal{I}}) = ({}^{\perp}dw\widetilde{\mathcal{I}}$, Complexes of injectives) in Ch(R).

$$\text{Ho}[\text{Ch}(R)] \simeq K(Inj) = dw\widetilde{\mathcal{I}}/\sim$$

The exact Inj model structure

$(\mathcal{W}_2, ex\widetilde{\mathcal{I}}) = ({}^{\perp}ex\widetilde{\mathcal{I}}$, Exact complexes of injectives) in Ch(R).

$$\text{Ho}[\text{Ch}(R)] \simeq K_{ex}(Inj) = ex\widetilde{\mathcal{I}}/\sim$$

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Gorenstein Injective Modules

Recall:

Definition: An R-module M is called **Gorenstein injective** if there exists an exact complex of injectives

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

with $M = \ker(I^0 \to I^1)$ and which remains exact after applying $\mathrm{Hom}_R(J, -)$ for any injective module J.

Recall:

Definition: An R-module M is called **Gorenstein AC-injective** if there exists an exact complex of injectives

$$\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

with $M = \ker(I^0 \to I^1)$ and which remains exact after applying $\mathrm{Hom}_R(J, -)$ for any absolutely clean module J.

Noetherian and coherent cases of Gorenstein AC-Injective

Noetherian Case: R Noetherian $\implies$ The Gorenstein AC-injective modules coincide with the usual Gorenstein injective modules.

(Reason) R Noetherian implies that the absolutely clean modules coincide with the injective modules.

Coherent Case: R coherent $\implies$ The Gorenstein AC-injective modules coincide with modules I call "Ding injective" modules.

(Reason) R coherent implies that the absolutely clean modules coincide with the absolutely pure modules.

Gorenstein AC-injective model structure on R-Mod

Theorem (Bravo/Gillespie/Hovey)

Let R be any ring and let $\mathcal{GI}$ denote the class of Gorenstein AC-injective modules. Set $\mathcal{W}_{inj} = {}^{\perp}\mathcal{GI}$. Then $\mathcal{M}_{inj} = (\mathcal{W}_{inj}, \mathcal{GI})$ is an injective cotorsion pair. That is,

- *$\mathcal{W}_{inj}$ is thick, and contains all projectives and injectives.*
- *$\mathcal{W}_{inj} \cap \mathcal{GI}$ is the class of injectives.*
- *$(\mathcal{W}_{inj}, \mathcal{GI})$ is complete, in fact, cogenerated by a set.*

Proof.

Explain how we built it on chain complexes... □

Corollaries of Gorenstein AC-injective model structure

Corollary: The "Gorenstein AC-injective model" on R-Mod.

$\mathcal{M}_{inj}$ is an injective abelian model structure on R-Mod whose fibrant objects are the Gorenstein AC-injectives. We get an equivalence of triangulated categories:

$$\text{Ho}[\mathcal{M}_{inj}] \simeq \mathcal{GI}/ \sim$$

Corollary: "Gorenstein AC-injective pre-envelopes" of R-modules.

Every R-module has a Gorenstein AC-injective pre-envelope. (These are fibrant replacements.)

Gorenstein Projective Modules

Recall:

Definition: An R-module M is called **Gorenstein projective** if there exists an exact complex of projectives

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \ker(P^0 \to P^1)$ and which remains exact after applying $\mathrm{Hom}_R(-, Q)$ for any projective module Q.

Recall:

Definition: An R-module M is called **Gorenstein AC-projective** if there exists an exact complex of projectives

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \ker(P^0 \to P^1)$ and which remains exact after applying $\mathrm{Hom}_R(-, Q)$ for any level module Q.

Theorem (Bravo/Gillespie/Hovey)

Let R be any ring and let $\mathcal{GP}$ denote the class of Gorenstein AC-projective modules. Set $\mathcal{W}_{prj} = \mathcal{GP}^{\perp}$. Then $\mathcal{M}_{prj} = (\mathcal{GP}, \mathcal{W}_{prj})$ is a projective cotorsion pair. That is,

- *$\mathcal{W}_{prj}$ is thick, and contains all projectives and injectives.*
- *$\mathcal{GP} \cap \mathcal{W}_{prj}$ is the class of projectives.*
- *$(\mathcal{GP}, \mathcal{W}_{prj})$ is complete, in fact, cogenerated by a set.*

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Summary:

- For an arbitrary ring R, we have two abelian model structures on R-Mod: $\mathcal{M}_{inj} = (\mathcal{W}_{inj}, \mathcal{GI})$ and $\mathcal{M}_{prj} = (\mathcal{GP}, \mathcal{W}_{prj})$.
- $\mathcal{W}_{inj}$ contains all projective and injective modules, in fact, it contains all abs. clean modules.
- $\mathcal{W}_{prj}$ contains all projective and injective modules, in fact, it contains all level modules.

Open Question: When do we have $\mathcal{W}_{inj} = \mathcal{W}_{prj}$?

Motivation: This would assign to R, a *unique* stable module cat:

$$\text{Stmod(R)} := \text{Ho}[\mathcal{M}_{inj}] = \text{Ho}[\mathcal{M}_{prj}]$$

.

Recall:

Definition: A ring R is a **Gorenstein ring** if R is Noetherian and there is an upper bound, say n, on the injective dimension of all flat modules.

Theorem (Fundamental Theorem)

Then the following are equivalent for any R-module M:

1. $fd(M) < \infty$.
2. $fd(M) \leq n$.
3. $id(M) < \infty$.
4. $id(M) \leq n$.

AC-Gorenstein rings

Recall:

Definition: A ring R is a **AC-Gorenstein ring** if R is any ring and there is an upper bound, say n, on the abs. clean dimension of all level modules.

Theorem (Fundamental Theorem)

Then the following are equivalent for any R-module M:

1. $ld(M) < \infty$.
2. $ld(M) \leq n$.
3. $ad(M) < \infty$.
4. $ad(M) \leq n$.

Noetherian and coherent cases of AC-Gorenstein rings

Noetherian Case: A Noetherian AC-Gorenstein ring is precisely a Iwanaga-Gorenstein ring.

(Reason) level = flat, and, absolutely clean = injective.

Coherent Case: A coherent AC-Gorenstein ring is precisely a Ding-Chen ring.

(Reason) level = flat, and, absolutely clean = absolutely pure.

Particular examples of AC-Gorenstein rings

Example

If R is commutative coherent and absolutely pure (as a module over itself), then the group ring RG is an AC-Gorenstein ring for any locally finite group G

Example

If R is commutative coherent and of finite absolutely pure dimension, then the group ring RG is an AC-Gorenstein ring for any finite group G.

Ques: When is a group algebra RG an AC-Gorenstein ring?

Examples

Let R be AC-Gorenstein. Then the graded ring $A := R[x]/(x^2)$ is also AC-Gorenstein. In some cases, we have $\text{Stmod(A)} \cong \mathcal{D}(R)$.

Main theorem on AC-Gorenstein rings

Theorem

Let R be an AC-Gorenstein ring and let $\mathcal{W}$ denote the class of all module of finite level (equivalently, absolutely clean) dimension. Then:

- $\mathcal{W}_{inj} = \mathcal{W} = \mathcal{W}_{prj}$
- $(\mathcal{GP}, \mathcal{W})$ *and* $(\mathcal{W}, \mathcal{GI})$ *are "balanced".*
- Stmod(R)$(:= Ho[\mathcal{M}_{inj}] = Ho[\mathcal{M}_{prj}])$ *is a compactly generated triangulated category.*

Thank You!