Oriented Exchange Graphs & Torsion Classes

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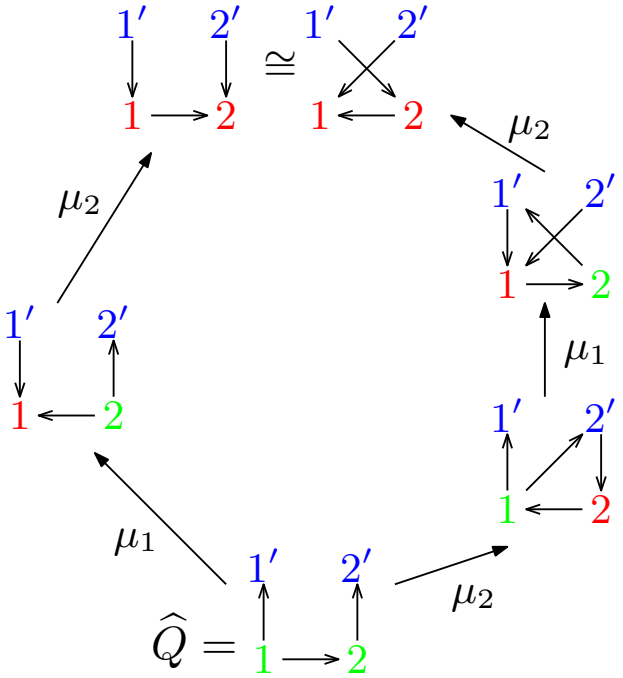
Outline

1. Oriented exchange graphs
2. Torsion classes & biclosed subcategories
3. Application: maximal green sequences

Oriented exchange graphs

Definition (Brüstle-Dupont-Pérotin)

The **oriented exchange graph** of Q, denoted $\overrightarrow{EG}(\widehat{Q})$, is the directed graph whose vertices are quivers mutation-equivalent to $\widehat{Q}$ and whose edges are $\overline{Q}_1 \to \mu_k \overline{Q}_1$ if and only if k is green in $\overline{Q}_1$.



The oriented exchange graph of $Q = 1 \to 2$ has **maximal green sequences** $(1, 2)$ and $(2, 1, 2)$.

Oriented exchange graphs

Maximal green sequences and oriented exchange graphs have connections with

- Donaldson-Thomas invariants and quantum dilogarithm identities [Keller, Joyce-Song, Kontsevich-Soibelman]
- BPS states in string theory [Alim-Cecotti-Cordova-Espahbodi-Rastogi-Vafa]
- Cambrian lattices (e.g Tamari lattices) [Reading 2006]

Torsion classes & biclosed subcategories

Where we are going

From a quiver Q that is mutation-equivalent to $1 \to 2 \to \cdots \to n$, one obtains a finite dimensional $\Bbbk$-algebra $\Lambda = \Bbbk Q/I$ (known as a **cluster-tilted algebra**).

$$\Lambda\text{-mod} \quad \simeq \quad \mathrm{rep}_{\Bbbk}(Q, I) := \left\{ \begin{array}{c} \textbf{representations} \text{ of} \\ Q \text{ satisfying } I \end{array} \right\}$$

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- *Goal:* use nice subcategories of Λ-mod to understand the poset structure of $\overrightarrow{EG}(\widehat{Q})$.

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Theorem (Brüstle–Yang, Ingalls–Thomas)

Let Q be mutation-equivalent to a Dynkin quiver. Then $\overrightarrow{EG}(\widehat{Q}) \cong \mathit{tors}(\Lambda)$ as posets where $\Lambda = \Bbbk Q/I$ is the cluster-tilted algebra associated to Q.

Torsion classes

Theorem (Butler-Ringel)

The indecomposable Λ-modules are parameterized by full, connected subquivers of Q that contain at most one arrow from each oriented cycle of Q.

Example ($Q = 1 \xrightarrow{\alpha} 2$)

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The **Auslander-Reiten quiver** of Λ-mod is

$$\Gamma(\Lambda\text{-mod}) \quad = \quad \begin{tikzcd} & \Bbbk \arrow[r, "1"] & \Bbbk \arrow[dr] & \\ 0 \arrow[r, "0"] & \Bbbk \arrow[ur, bend left] & & \Bbbk \arrow[r, "0"] & 0. \end{tikzcd}$$

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We use $\Gamma(\Lambda\text{-mod})$ to describe the torsion classes of Λ.

Torsion classes

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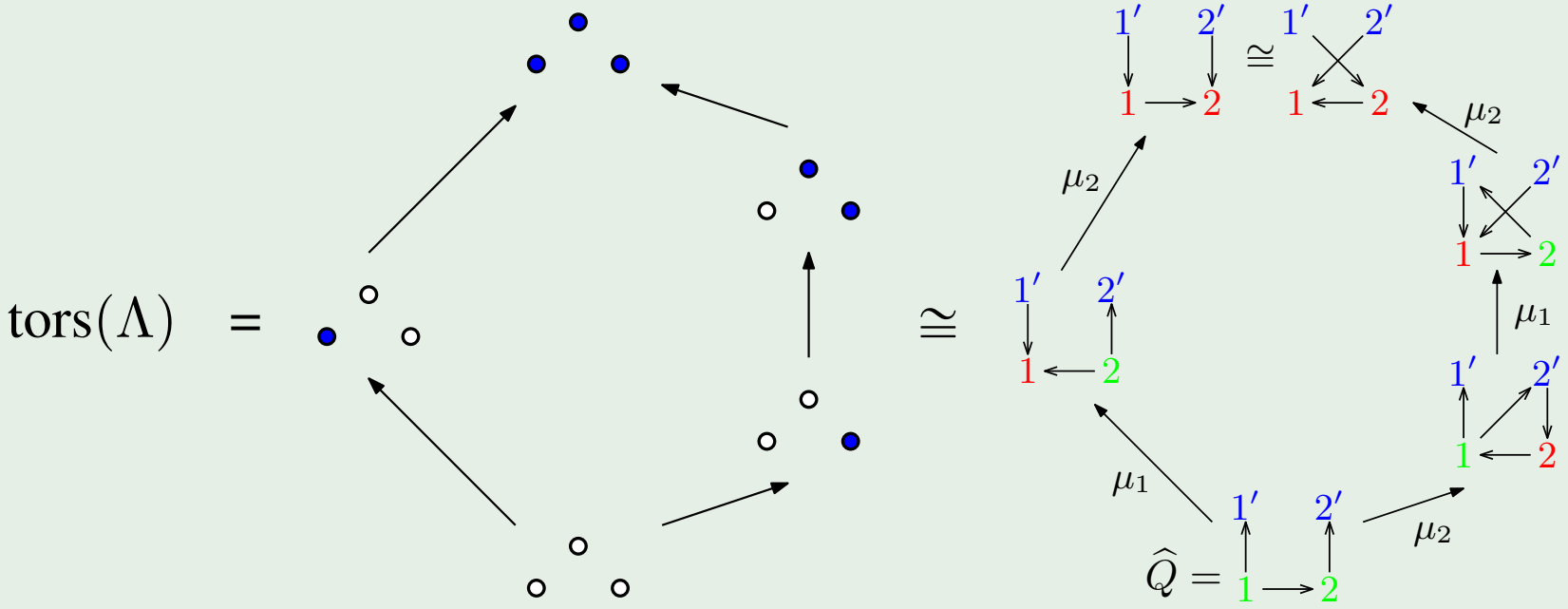

$\mathrm{tors}(\Lambda)$:= torsion classes of Λ ordered by inclusion

A full, additive subcategory $\mathcal{T}$ of Λ-mod is a **torsion class** of Λ if it is

a) **extension closed** : if $X, Y \in \mathcal{T}$ and one has an exact sequence $0 \to X \to Z \to Y \to 0$, then $Z \in \mathcal{T}$,

b) **quotient closed** : $X \in \mathcal{T}$ and $X \twoheadrightarrow Z$ implies $Z \in \mathcal{T}$.

Torsion classes

The partially ordered set $\mathrm{tors}(\Lambda)$ is a **lattice** (i.e. any two torsion classes $\mathcal{T}_1, \mathcal{T}_2 \in \mathrm{tors}(\Lambda)$ have a **join** (resp. **meet**), denoted $\mathcal{T}_1 \vee \mathcal{T}_2$ (resp. $\mathcal{T}_1 \wedge \mathcal{T}_2$)).

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Let Λ be a finite dimensional $\Bbbk$-algebra and let $\mathcal{T}_1, \mathcal{T}_2 \in \mathrm{tors}(\Lambda)$. Then

i) $\mathcal{T}_1 \wedge \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2$,

ii) $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{F}ilt(\mathcal{T}_1, \mathcal{T}_2)$ *where* $\mathcal{F}ilt(\mathcal{T}_1, \mathcal{T}_2)$ *consists of objects* X *with a filtration* $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ *such that* X_j/X_{j-1} *belongs to* $\mathcal{T}_1$ *or* $\mathcal{T}_2$. *[G.–McConville]*

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Theorem (G.–McConville)

If Q is mutation-equivalent to a Dynkin quiver, then $\overrightarrow{EG}(\widehat{Q}) \cong \mathrm{tors}(\Lambda)$ is a ***semidistributive lattice*** *(i.e. $\mathcal{T}_1 \wedge \mathcal{T}_3 = \mathcal{T}_2 \wedge \mathcal{T}_3$ implies that $(\mathcal{T}_1 \vee \mathcal{T}_2) \wedge \mathcal{T}_3 = \mathcal{T}_1 \wedge \mathcal{T}_3$ and the dual statement holds).*

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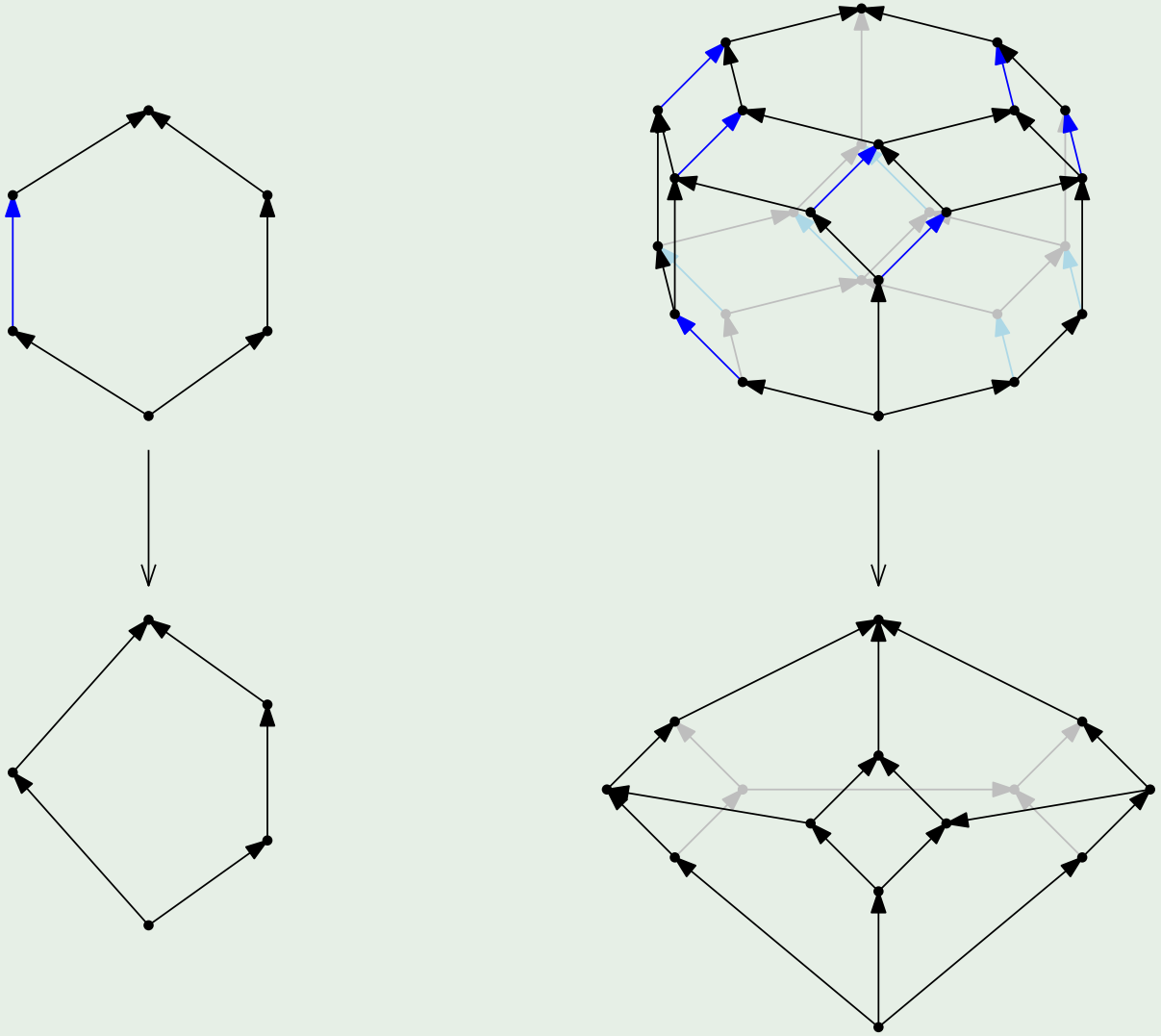
Goal: Realize $\mathrm{tors}(\Lambda)$ as a **quotient** of a lattice with nice properties so that $\mathrm{tors}(\Lambda)$ will inherit these nice properties.

Torsion classes

Example

A lattice quotient map $\pi_{\downarrow} : L \to L/\sim$ is a surjective map of lattices.



Biclosed subcategories

Now we assume that Q is mutation-equivalent to $1 \to 2 \to \cdots \to n$.

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$\mathcal{BIC}(Q)$:= biclosed subcategories of Λ-mod ordered by inclusion

A full, additive subcategory $\mathcal{B}$ of Λ-mod is **biclosed** if

a) $\mathcal{B} = \mathrm{add}(\bigoplus_{i=1}^{k} X_i)$ for some set of indecomposables $\{X_i\}_{i=1}^{k}$ (here $\mathrm{add}(\bigoplus_{i=1}^{k} X_i)$ consists of objects $\bigoplus_{i=1}^{k} X_i^{m_i}$ where $m_i \geqslant 0$),

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b) $\mathcal{B}$ is **weakly extension closed**: if $0 \to X_1 \to X \to X_2 \to 0$ is an exact sequence where X_1, X_2, X are indecomposable and $X_1, X_2 \in \mathcal{B}$, then $X \in \mathcal{B}$,

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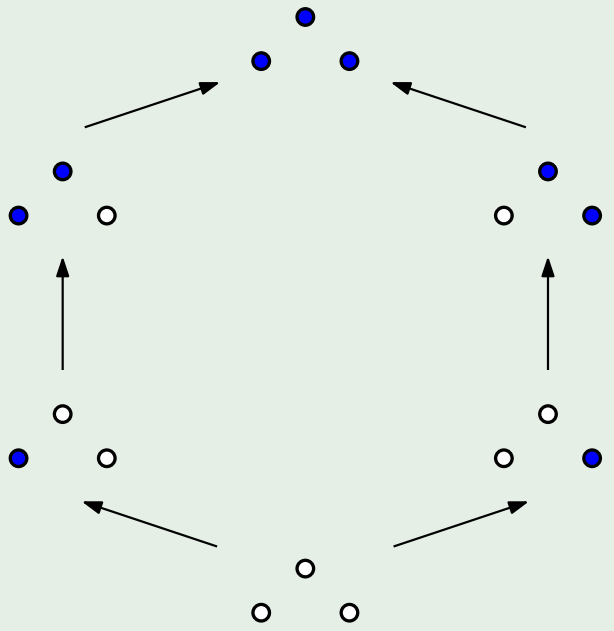
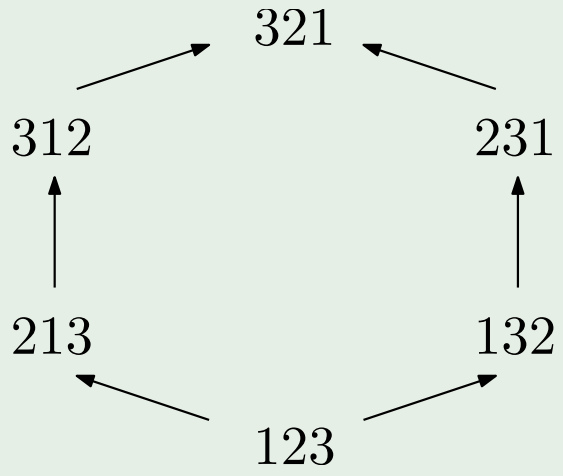


$\mathcal{BIC}(Q)$

$\cong$



weak order on elements of $\mathfrak{S}_3$

The family of lattices of the form $\mathcal{BIC}(Q)$ *properly* contains the lattices isomorphic to the weak order on $\mathfrak{S}_n$.

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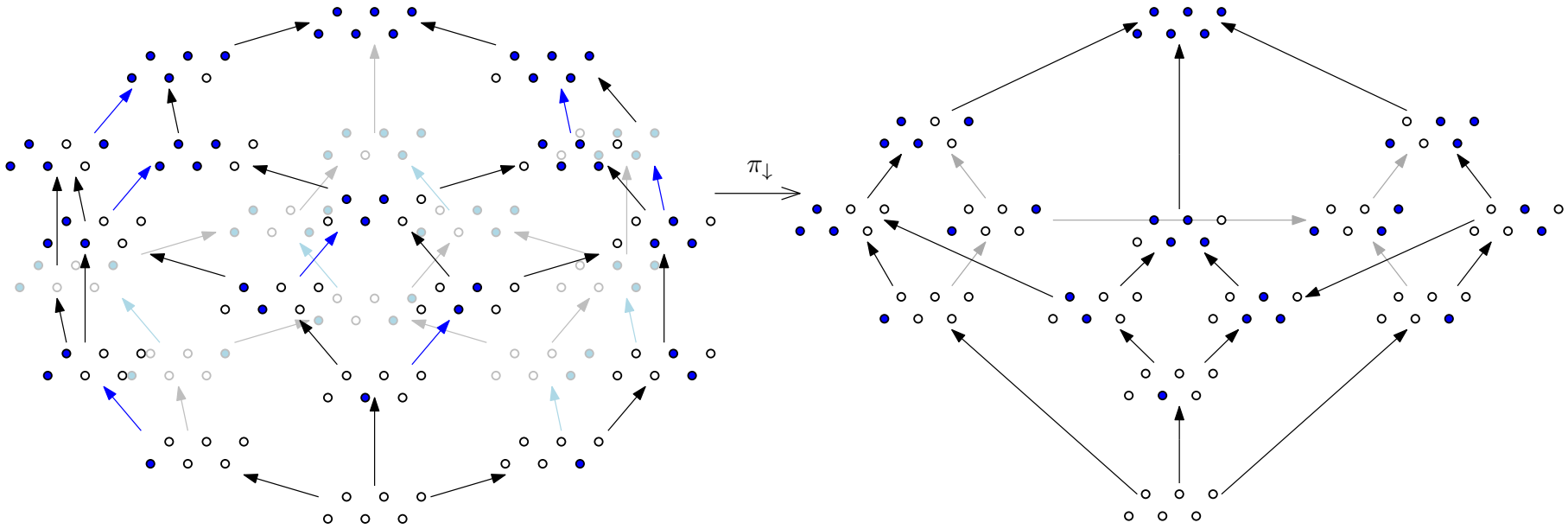

Theorem (G.– McConville)

Let $\mathcal{B} = add(\oplus_{i=1}^{k} X_i) \in \mathcal{BIC}(Q)$ *and let*

$$\pi_\downarrow(\mathcal{B}) := add(\oplus_{j=1}^{\ell} X_{i_j} : X_{i_j} \twoheadrightarrow Y \implies Y \in \mathcal{B}).$$

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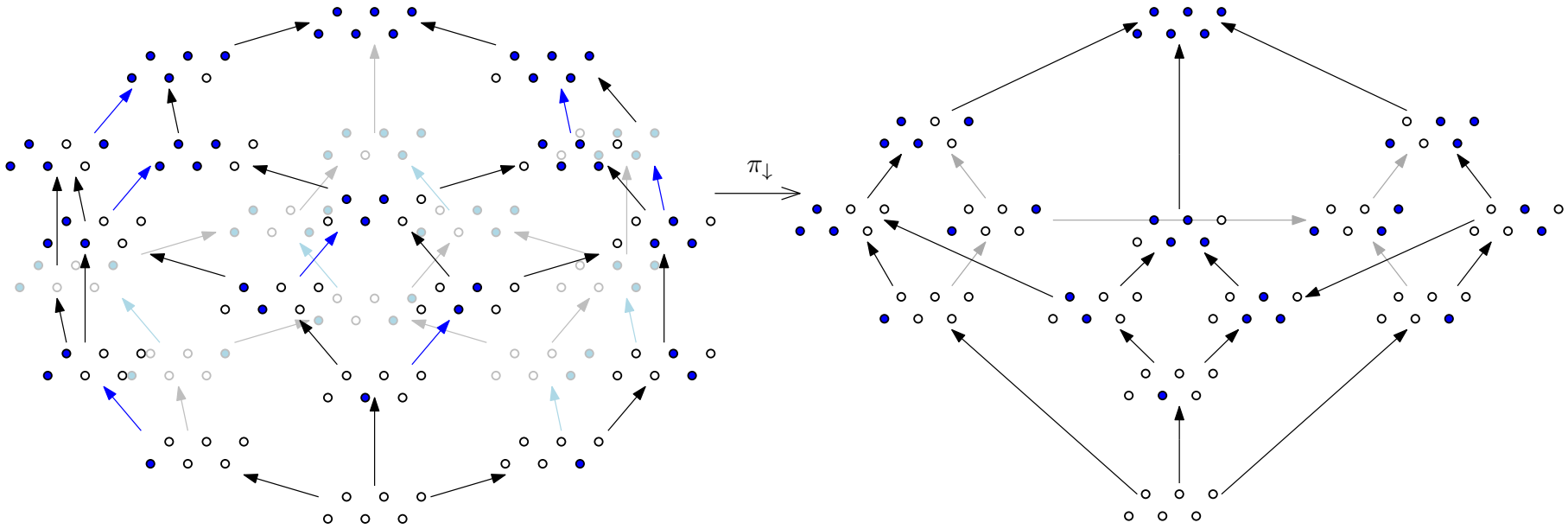

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Then $\pi_\downarrow : \mathcal{BIC}(Q) \to tors(\Lambda)$ *is a lattice quotient map.*

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Theorem (Caspard–Le Conte de Poly-Barbut–Morvan 2004, Reading (proves polygonality) in forthcoming book)

The weak order on the symmetric group (in fact, on any finite Coxeter group) is semidistributive, congruence-uniform, and polygonal.

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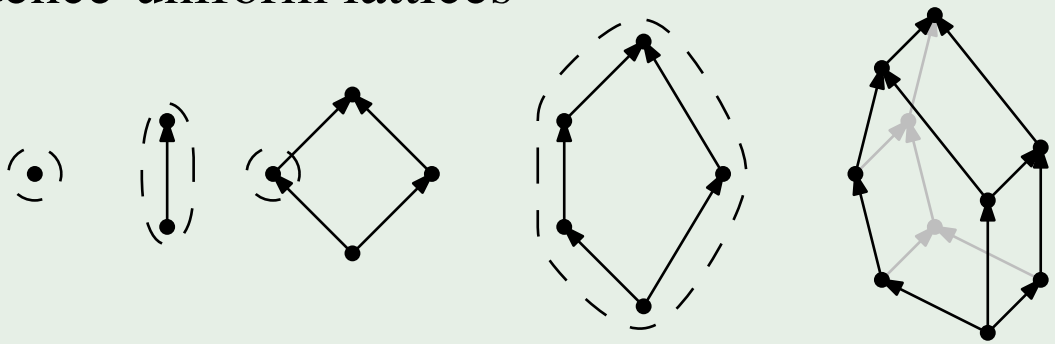
Example

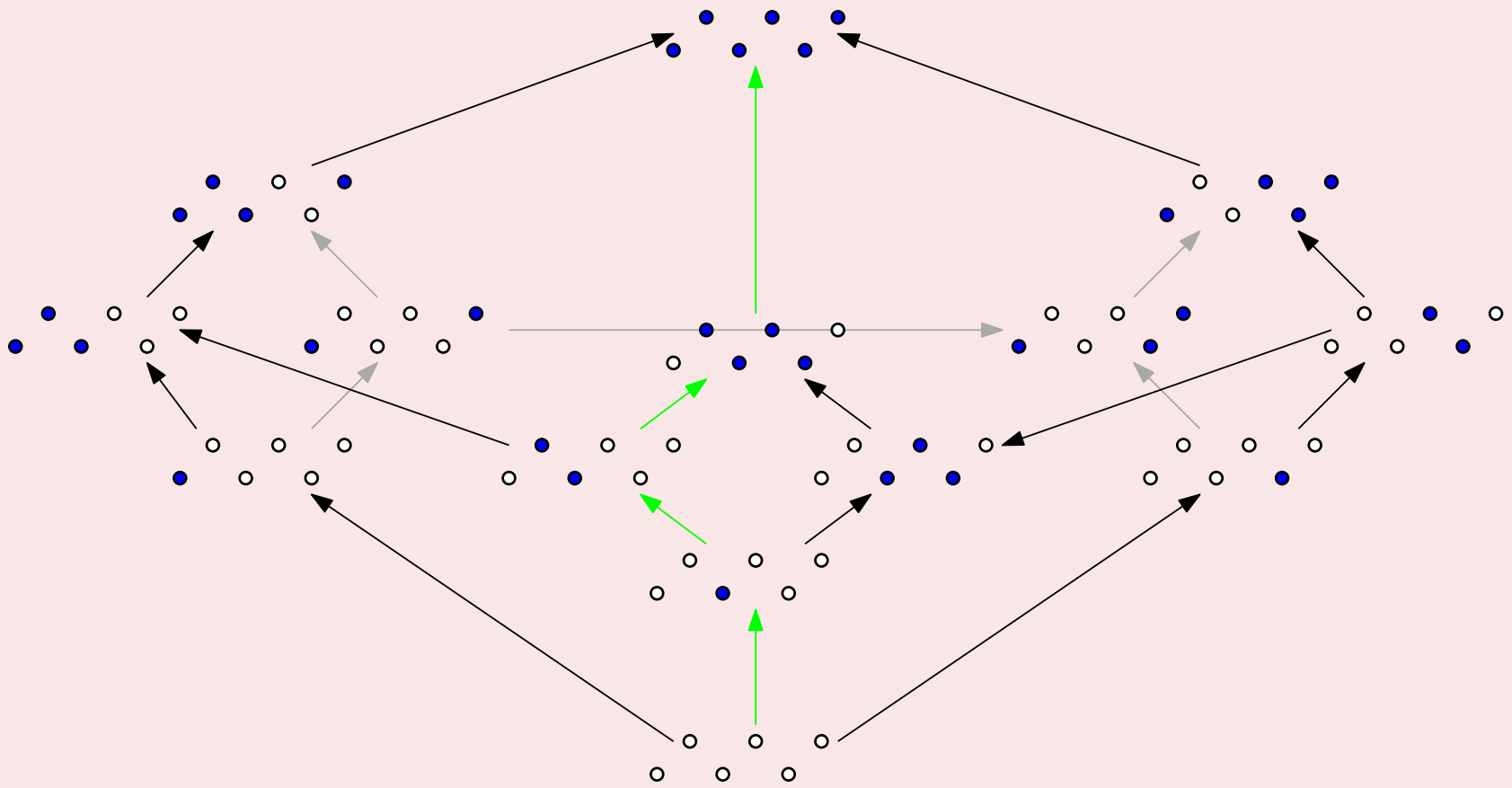
Some congruence-uniform lattices



Application: maximal green sequences

Example

The maximal green sequences of Q are connected by **polygonal flips**.



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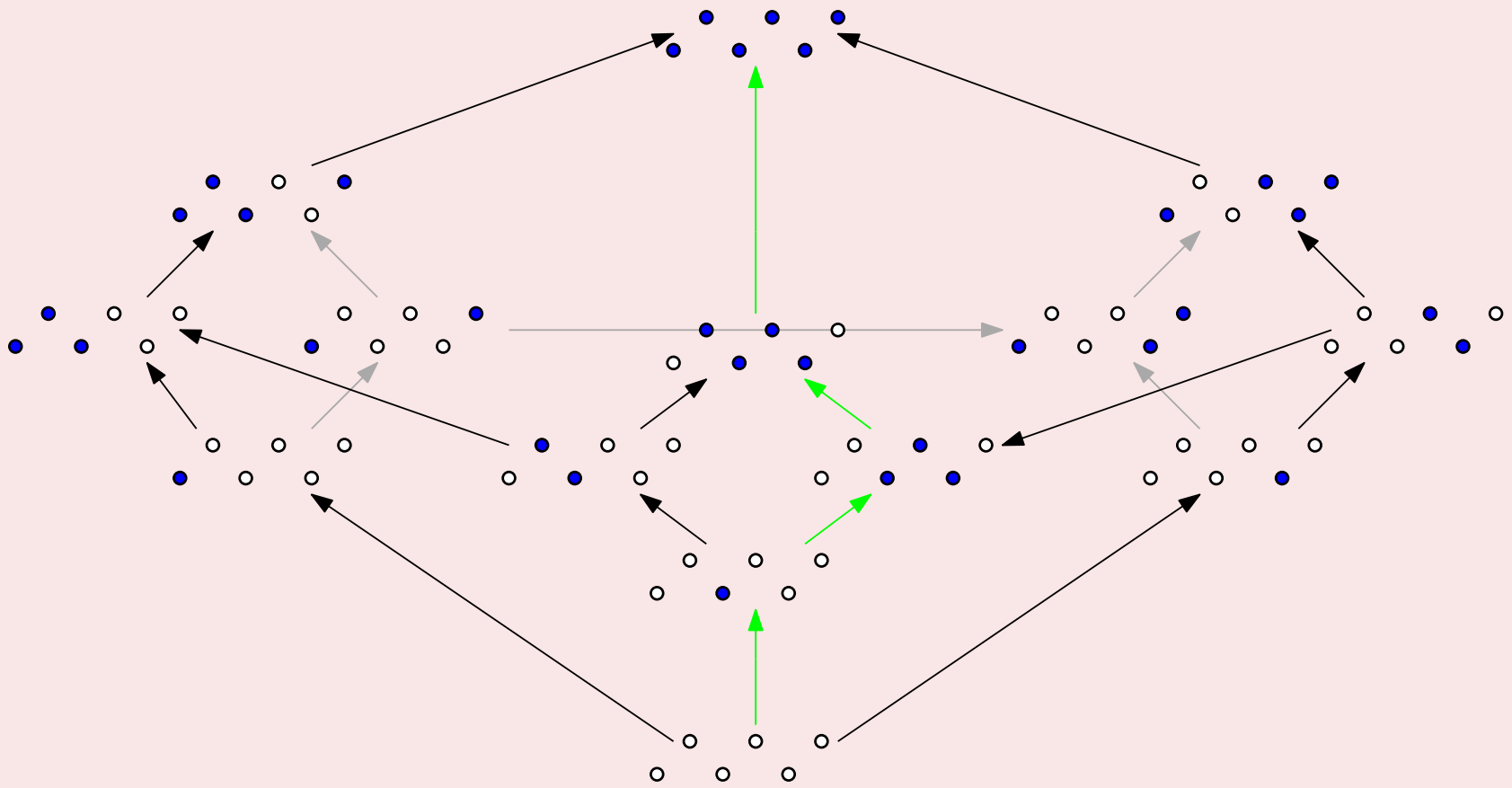

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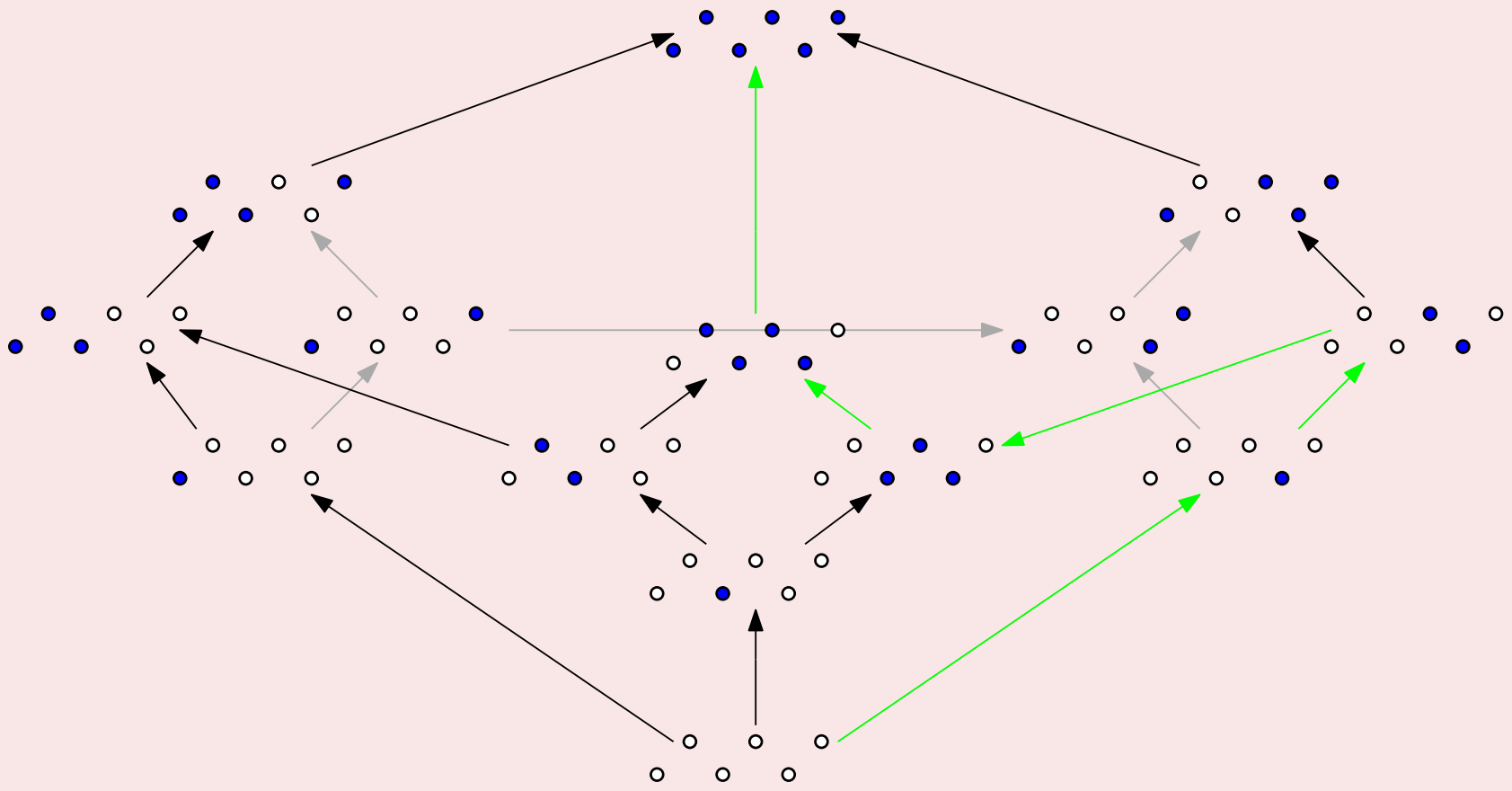

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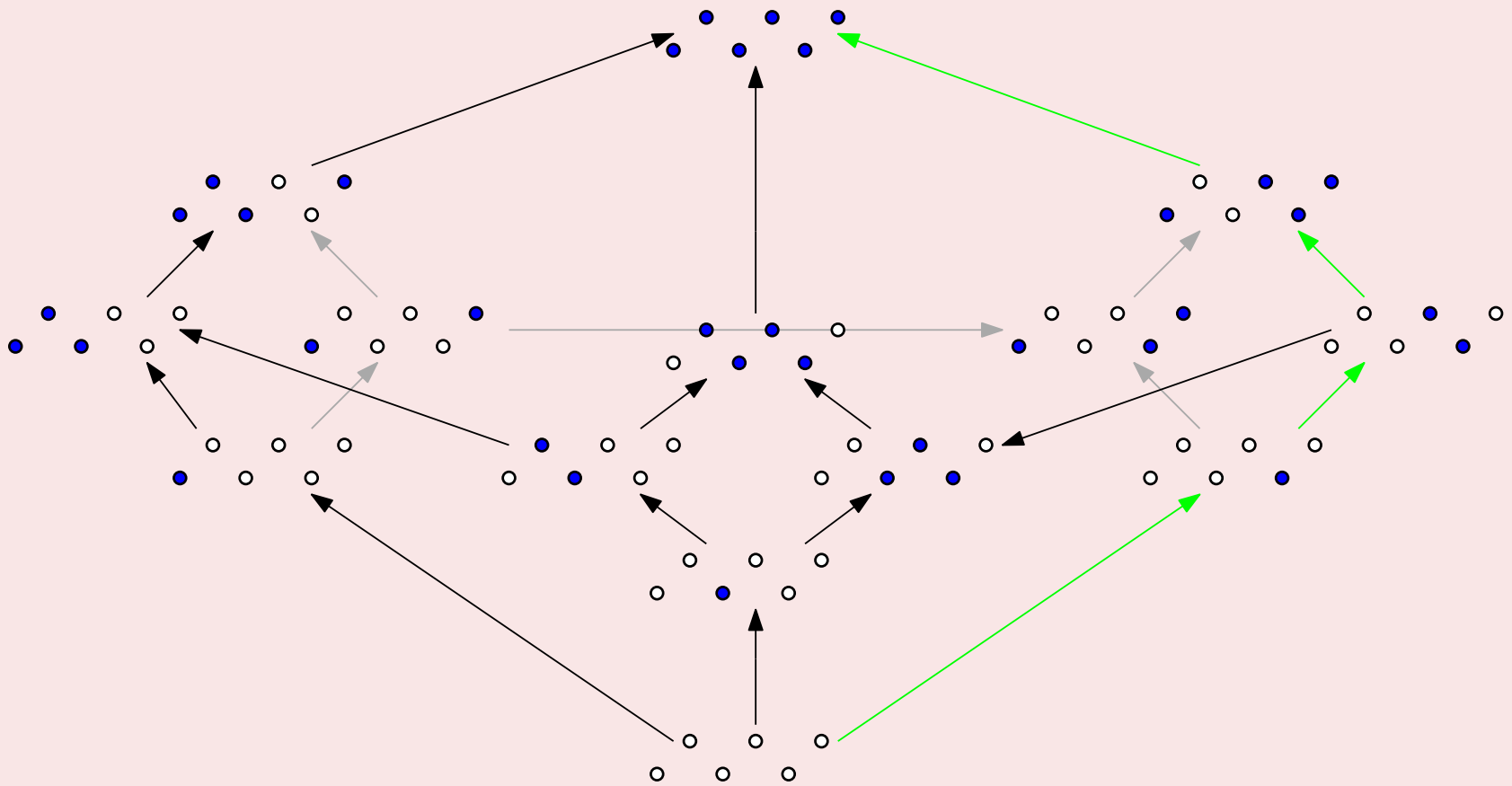

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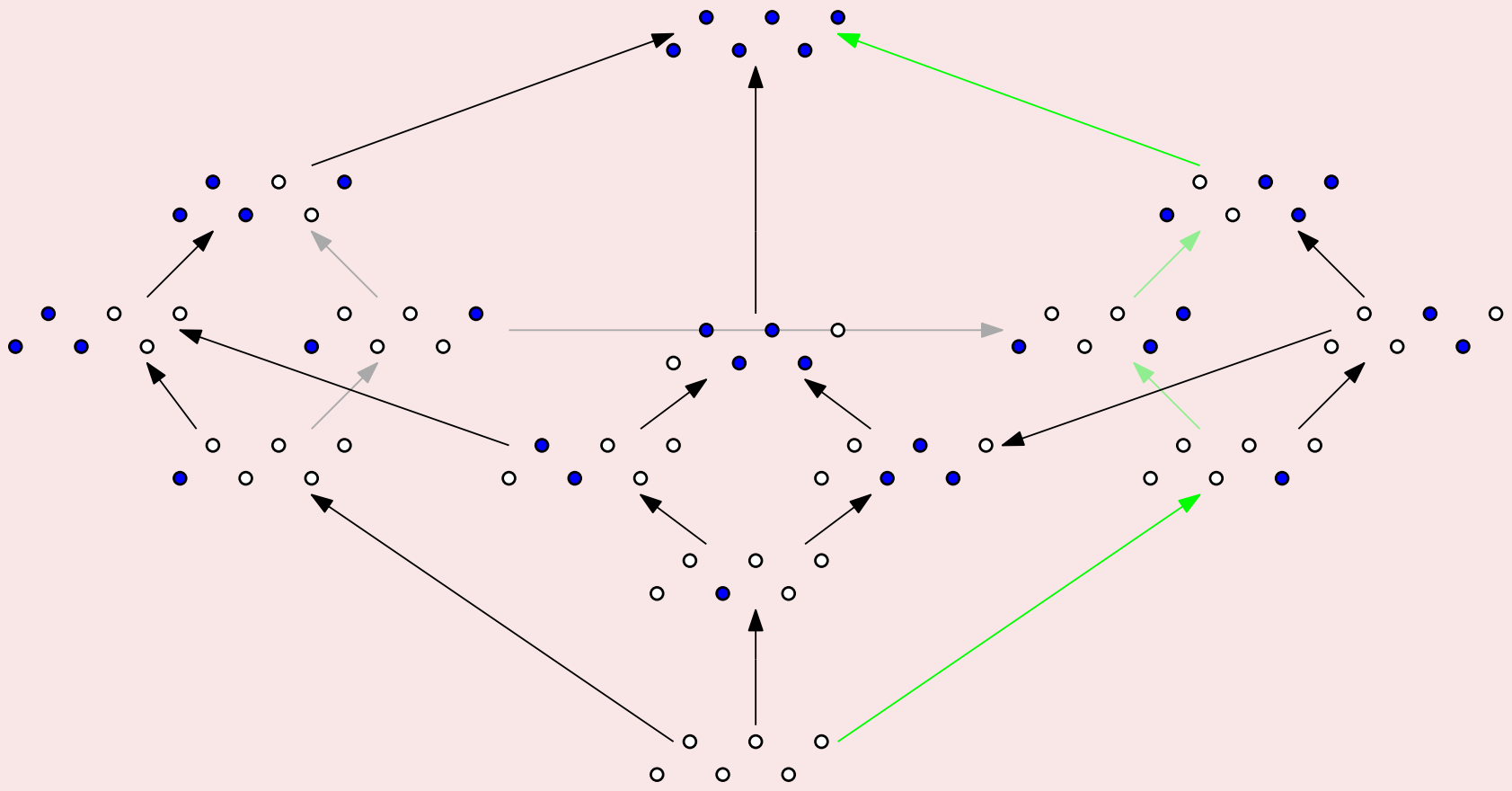

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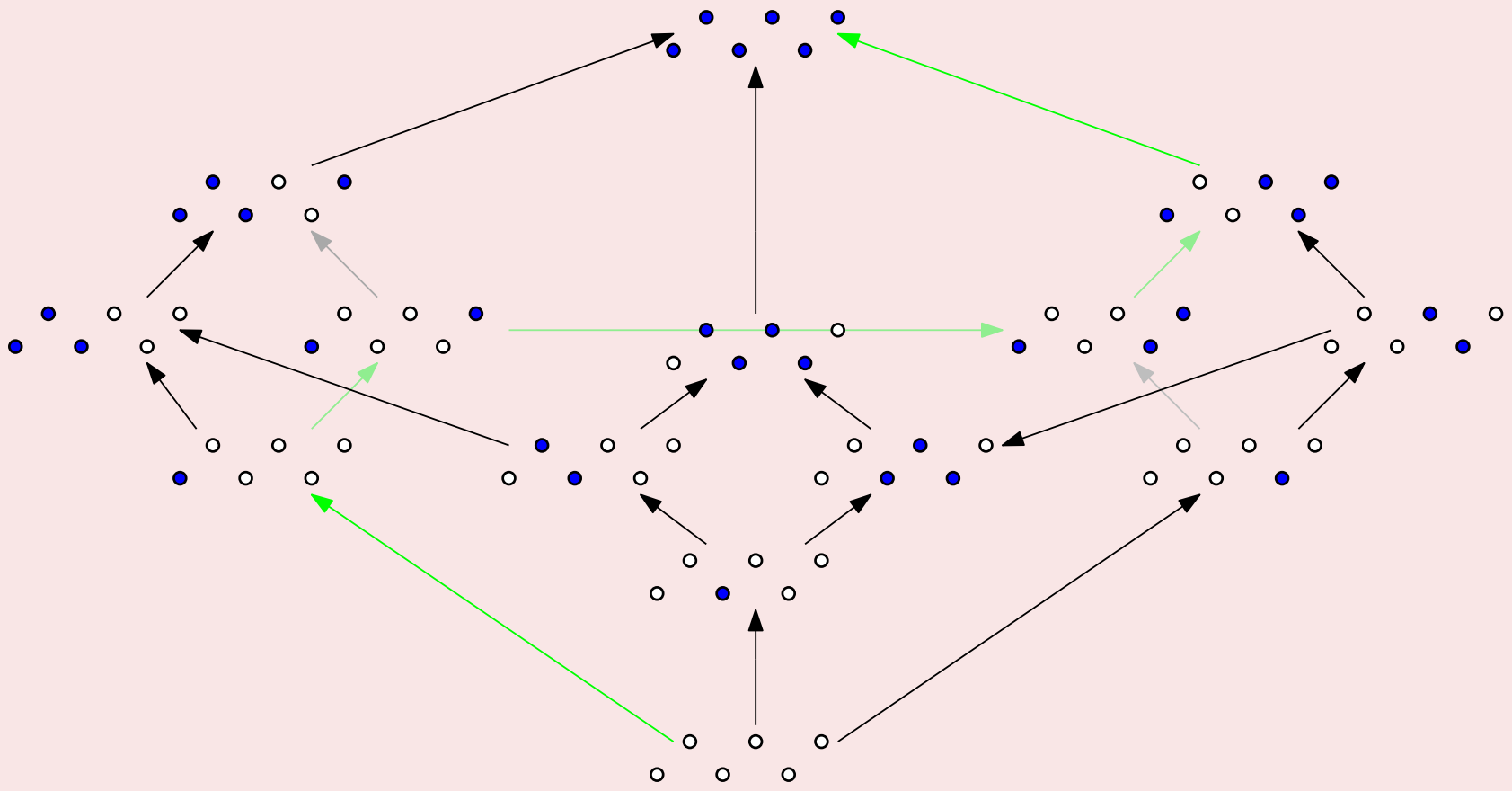

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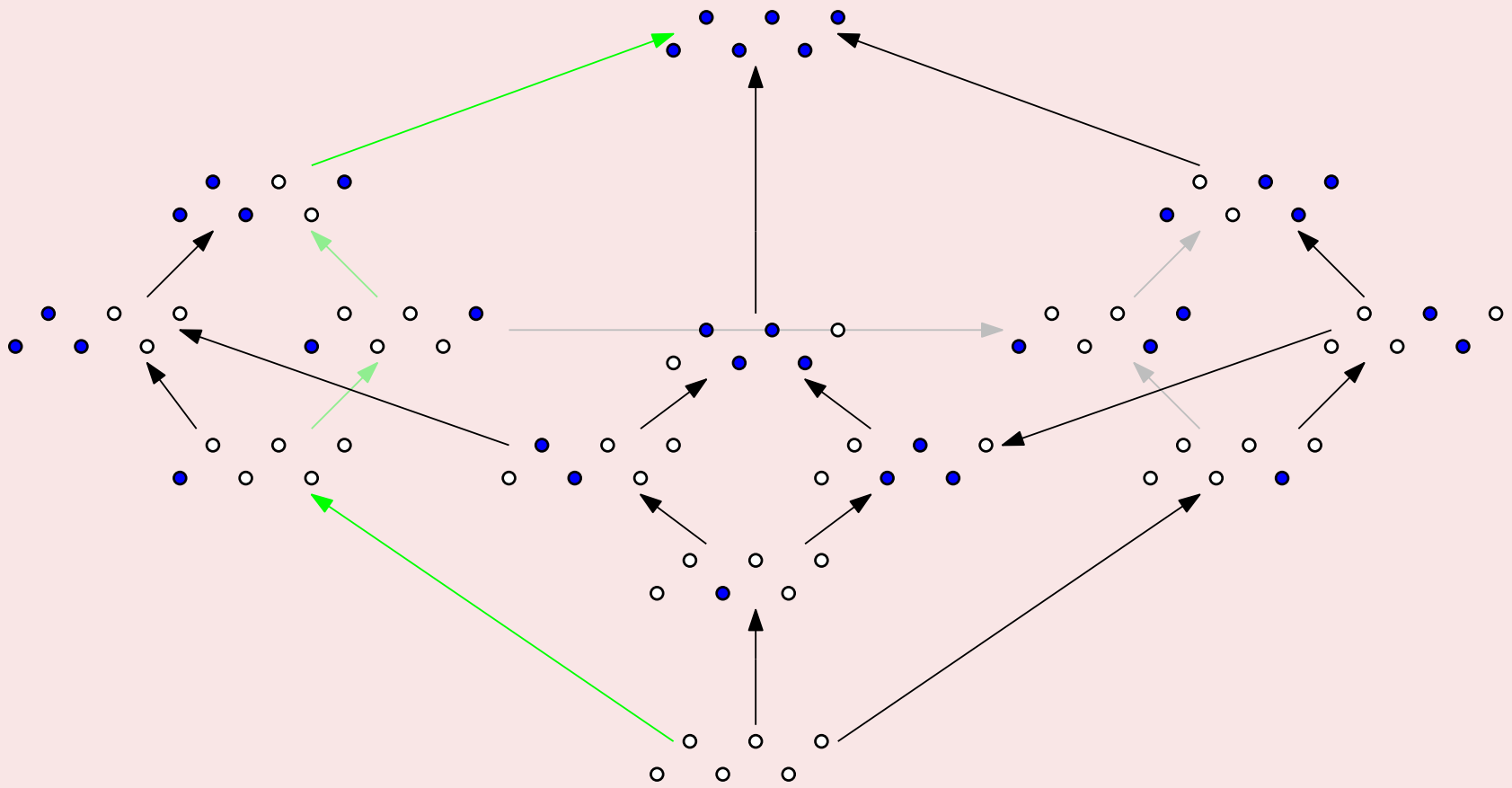

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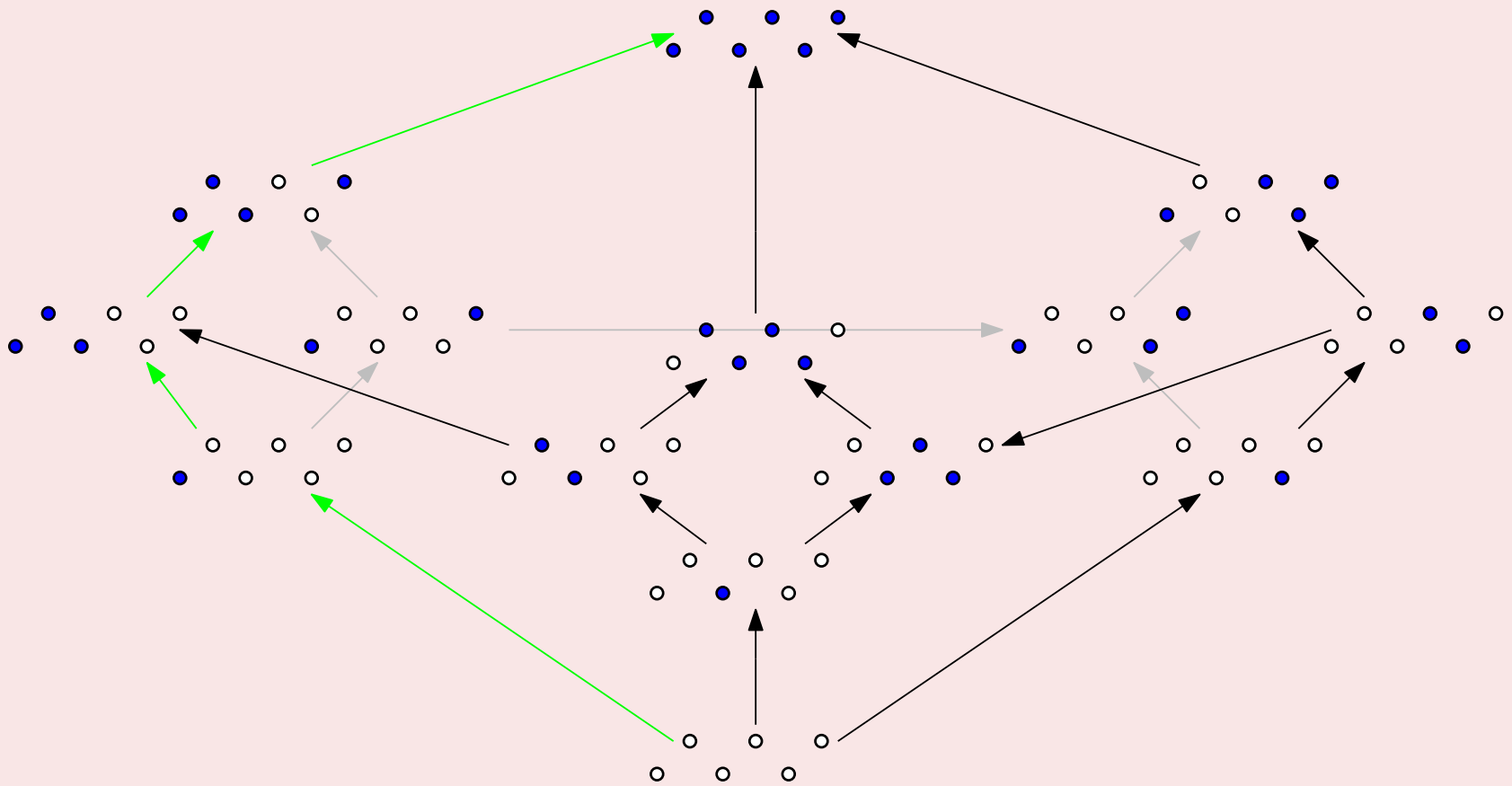

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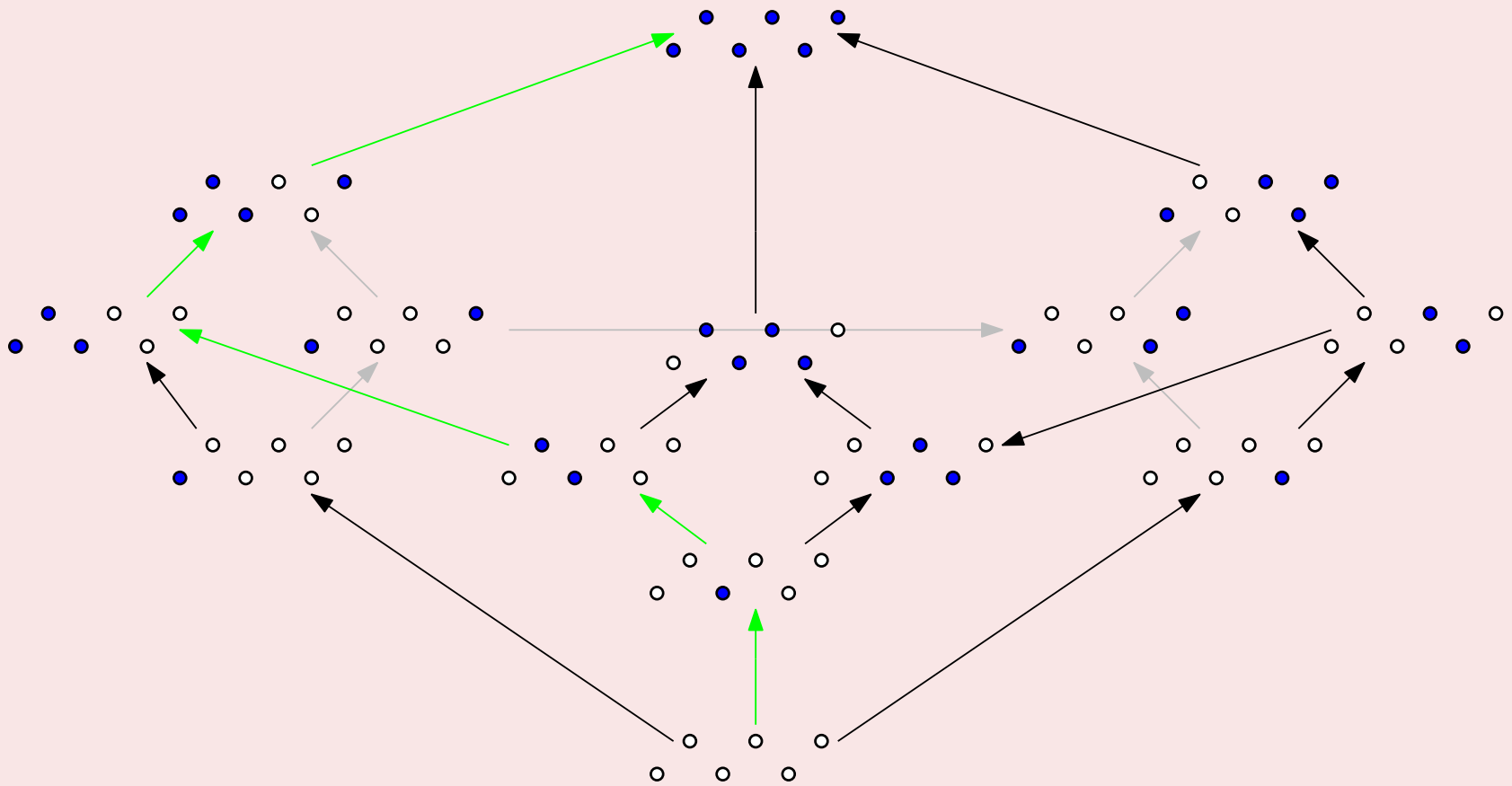

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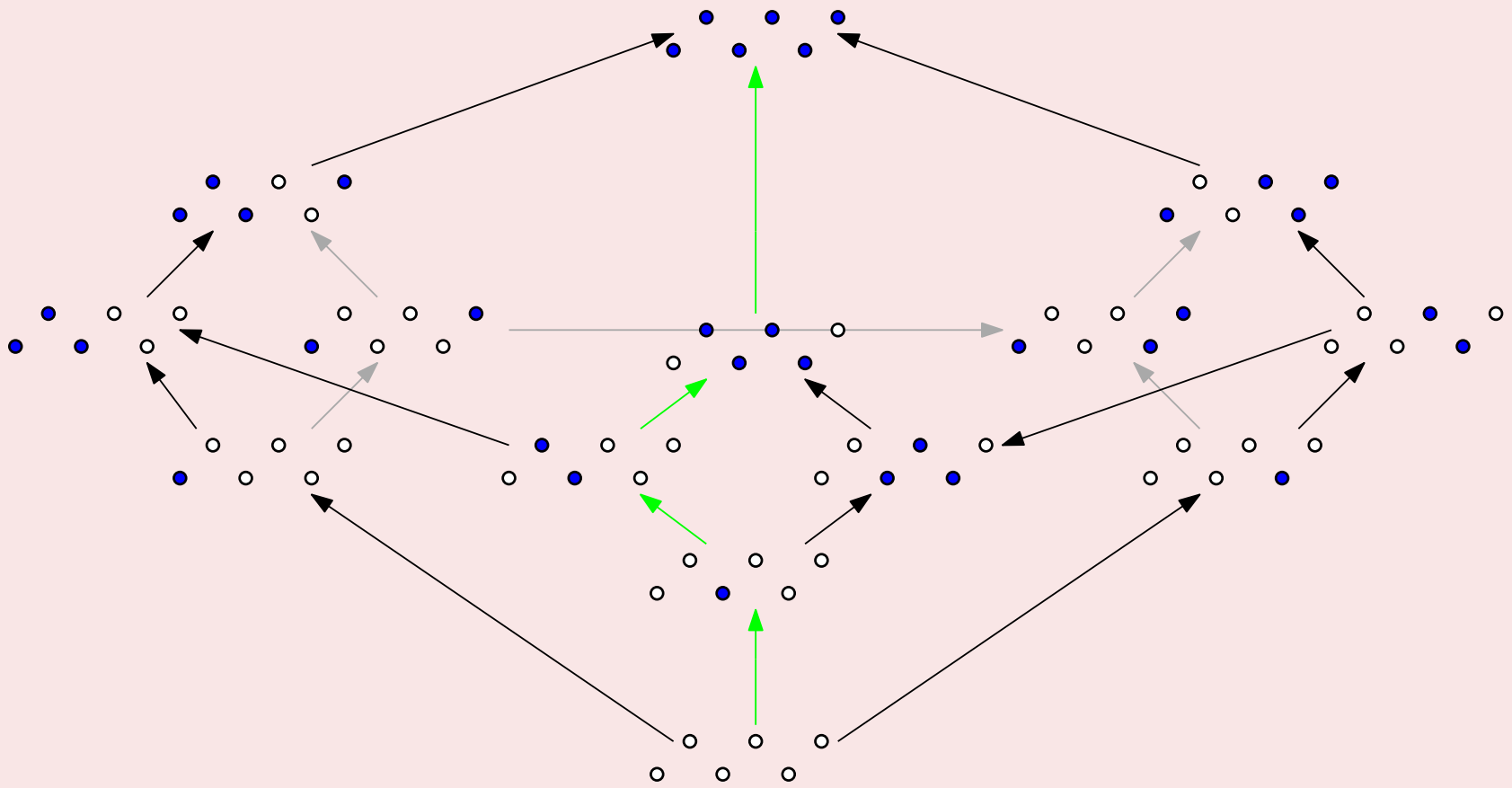

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Theorem (G.–McConville)

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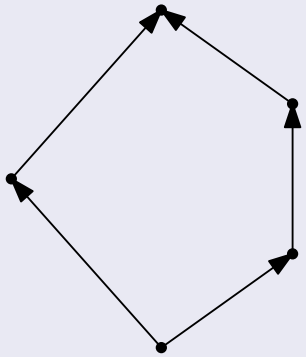

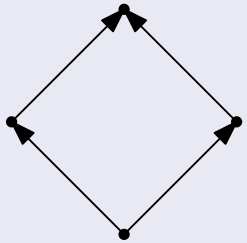

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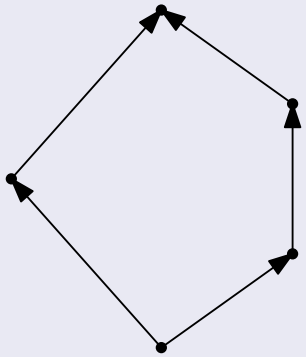

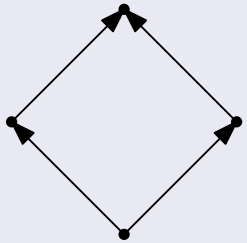

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Corollary (Conjectured by Brüstle-Dupont-Pérotin for any quiver Q)

If Q is mutation-equivalent to $1 \to 2 \to \cdots \to n$, the set of lengths of the maximal green sequences of Q is of the form $\{\ell_{min}, \ell_{min}+1, \ldots, \ell_{max}-1, \ell_{max}\}$ where

$$\ell_{min} := \textit{length of the shortest maximal green sequence of } Q,$$

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Thanks!