Oriented Exchange Graphs & Torsion Classes

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Outline

1. Oriented exchange graphs
2. Torsion classes & biclosed subcategories
3. Application: maximal green sequences
Oriented exchange graphs

Definition (Brüstle-Dupont-Pérotin)

The **oriented exchange graph** of \( Q \), denoted \( \overrightarrow{EG}(\hat{Q}) \), is the directed graph whose vertices are quivers mutation-equivalent to \( \hat{Q} \) and whose edges are \( \overline{Q}_1 \rightarrow \mu_k \overline{Q}_1 \) if and only if \( k \) is **green** in \( \overline{Q}_1 \).

The oriented exchange graph of \( Q = 1 \rightarrow 2 \) has **maximal green sequences** \((1, 2)\) and \((2, 1, 2)\).
Maximal green sequences and oriented exchange graphs have connections with

- Donaldson-Thomas invariants and quantum dilogarithm identities [Keller, Joyce-Song, Kontsevich-Soibelman]
- BPS states in string theory [Alim-Cecotti-Cordova-Espahbodi-Rastogi-Vafa]
- Cambrian lattices (e.g. Tamari lattices) [Reading 2006]
Where we are going

From a quiver $Q$ that is mutation-equivalent to $1 \to 2 \to \cdots \to n$, one obtains a finite dimensional $k$-algebra $\Lambda = kQ/I$ (known as a cluster-tilted algebra).

$$\Lambda\text{-mod} \simeq \text{rep}_k(Q, I) := \left\{ \text{representations of } Q \text{ satisfying } I \right\}$$
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- **Goal:** use nice subcategories of $\Lambda\text{-mod}$ to understand the poset structure of $\text{EG}(\hat{Q})$. 

Theorem (Brüstle–Yang, Ingalls–Thomas)

Let $Q$ be mutation-equivalent to a Dynkin quiver. Then $\text{EG}(\hat{Q})$–tors $\Lambda_q$ as posets where $\Lambda_q = kQ/I$ is the cluster-tilted algebra associated to $Q$. 


Where we are going

From a quiver $Q$ that is mutation-equivalent to $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$, one obtains a finite dimensional $\mathbb{k}$-algebra $\Lambda = \mathbb{k}Q/I$ (known as a cluster-tilted algebra).

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\Lambda\text{-mod} \cong \text{rep}_\mathbb{k}(Q, I) := \left\{ \text{representations of } Q \text{ satisfying } I \right\}
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Theorem (Brüstle–Yang, Ingalls–Thomas)

*Let $Q$ be mutation-equivalent to a Dynkin quiver. Then $\overrightarrow{EG}(\hat{Q}) \cong \text{tors}(\Lambda)$ as posets where $\Lambda = \mathbb{k}Q/I$ is the cluster-tilted algebra associated to $Q$.***
Theorem (Butler-Ringel)

*The indecomposable* $\Lambda$-modules are parameterized by full, connected subquivers of $Q$ that contain at most one arrow from each oriented cycle of $Q$.

Example ($Q = 1 \xrightarrow{\alpha} 2$)

The cluster-tilted algebra associated to $Q$ is $\Lambda = \mathbb{k}Q$. 


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The Auslander-Reiten quiver of $\Lambda$-mod is

\[
\Gamma(\Lambda\text{-mod}) = \\
\begin{array}{cccccc}
\mathbb{k} & 1 & \mathbb{k} \\
\uparrow & & \downarrow \\
0 & \mathbb{k} & 0 & \mathbb{k} & 0 & 0.
\end{array}
\]
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Example ($Q = 1 \xrightarrow{\alpha} 2$)

The cluster-tilted algebra associated to $Q$ is $\Lambda = kQ$. The **Auslander-Reiten quiver** of $\Lambda$-mod is

$$
\begin{align*}
\Gamma(\Lambda\text{-mod}) &= \\
\text{  } & \\
\text{  } & \\
\text{  } & \\
-k & \xrightarrow{0} k & k & \xrightarrow{1} k \\
& \downarrow & & \downarrow \\
0 & \xrightarrow{0} \text{  } & k & \xrightarrow{0} \text{  } & 0.
\end{align*}
$$

We use $\Gamma(\Lambda\text{-mod})$ to describe the **torsion classes** of $\Lambda$. 


Torsion classes

Example \((Q = 1 \xrightarrow{\alpha} 2)\)

\[ \text{tors}(\Lambda) = \mathbb{Q} \]

\[ \hat{Q} = \mathbb{Q} \]

\[ \mu_1, \mu_2 \]

\[ 1 \xrightarrow{\mu_1} 1' \]
\[ 2 \xrightarrow{\mu_2} 2' \]

\[ 1' \xrightarrow{\mu_2} 1 \]
\[ 2' \xrightarrow{\mu_1} 2 \]

\[ \text{tors}(\Lambda) := \text{torsion classes of } \Lambda \text{ ordered by inclusion} \]

A full, additive subcategory \(\mathcal{T}\) of \(\Lambda\)-mod is a **torsion class** of \(\Lambda\) if it is

a) **extension closed**: if \(X, Y \in \mathcal{T}\) and one has an exact sequence
\[ 0 \to X \to Z \to Y \to 0, \text{ then } Z \in \mathcal{T}, \]

b) **quotient closed**: \(X \in \mathcal{T}\) and \(X \twoheadrightarrow Z\) implies \(Z \in \mathcal{T}\).
Torsion classes

The partially ordered set $\text{tors}(\Lambda)$ is a **lattice** (i.e. any two torsion classes $\mathcal{T}_1, \mathcal{T}_2 \in \text{tors}(\Lambda)$ have a **join** (resp. **meet**), denoted $\mathcal{T}_1 \vee \mathcal{T}_2$ (resp. $\mathcal{T}_1 \wedge \mathcal{T}_2$)).
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**Lemma**

*Let $\Lambda$ be a finite dimensional $k$-algebra and let $\mathcal{T}_1, \mathcal{T}_2 \in \text{tors}(\Lambda)$. Then*
Torsion classes

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**Lemma**

*Let $\Lambda$ be a finite dimensional $k$-algebra and let $\mathcal{T}_1, \mathcal{T}_2 \in \text{tors}(\Lambda)$. Then*

1. $\mathcal{T}_1 \land \mathcal{T}_2 = \mathcal{T}_1 \cap \mathcal{T}_2$,
2. $\mathcal{T}_1 \lor \mathcal{T}_2 = \mathcal{Filt}(\mathcal{T}_1, \mathcal{T}_2)$ where $\mathcal{Filt}(\mathcal{T}_1, \mathcal{T}_2)$ consists of objects $X$ with a filtration $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that $X_j/X_{j-1}$ belongs to $\mathcal{T}_1$ or $\mathcal{T}_2$. [G.–McConville]
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**Lemma**

Let \( \Lambda \) be a finite dimensional \( k \)-algebra and let \( T_1, T_2 \in \text{tors}(\Lambda) \). Then

i) \( T_1 \land T_2 = T_1 \cap T_2 \),

ii) \( T_1 \lor T_2 = \text{Filt}(T_1, T_2) \) where \( \text{Filt}(T_1, T_2) \) consists of objects \( X \) with a filtration \( 0 = X_0 \subset X_1 \subset \cdots \subset X_n = X \) such that \( X_j/X_{j-1} \) belongs to \( T_1 \) or \( T_2 \). [G.–McConville]

**Theorem (G.–McConville)**

If \( Q \) is mutation-equivalent to a Dynkin quiver, then \( \text{EG}(\hat{Q}) \cong \text{tors}(\Lambda) \) is a semidistributive lattice (i.e. \( T_1 \land T_3 = T_2 \land T_3 \) implies that \( (T_1 \lor T_2) \land T_3 = T_1 \land T_3 \) and the dual statement holds).
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The partially ordered set \( \text{tors}(\Lambda) \) is a \textbf{lattice} (i.e. any two torsion classes \( T_1, T_2 \in \text{tors}(\Lambda) \) have a \textbf{join} (resp. \textbf{meet}), denoted \( T_1 \lor T_2 \) (resp. \( T_1 \land T_2 \)).

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**Theorem (G.–McConville)**

If \( Q \) is mutation-equivalent to a Dynkin quiver, then \( \overline{\text{EG}}(\widehat{Q}) \cong \text{tors}(\Lambda) \) is a \textbf{semidistributive lattice} (i.e. \( T_1 \land T_3 = T_2 \land T_3 \) implies that \( (T_1 \lor T_2) \land T_3 = T_1 \land T_3 \) and the dual statement holds).

**Goal:** Realize \( \text{tors}(\Lambda) \) as a \textbf{quotient} of a lattice with nice properties so that \( \text{tors}(\Lambda) \) will inherit these nice properties.
Torsion classes

Example

A lattice quotient map $\pi : L \rightarrow L/\sim$ is a surjective map of lattices.
Biclosed subcategories

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$\mathcal{BIC}(Q) :=$ biclosed subcategories of $\Lambda$-mod ordered by inclusion

A full, additive subcategory $\mathcal{B}$ of $\Lambda$-mod is biclosed if

a) $\mathcal{B} = \text{add}(\bigoplus_{i=1}^{k} X_i)$ for some set of indecomposables $\{X_i\}_{i=1}^{k}$
   (here $\text{add}(\bigoplus_{i=1}^{k} X_i)$ consists of objects $\bigoplus_{i=1}^{k} X_i^{m_i}$ where $m_i \geq 0$),
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b) $\mathcal{B}$ is **weakly extension closed**: if $0 \to X_1 \to X \to X_2 \to 0$ is an exact sequence where $X_1, X_2, X$ are indecomposable and $X_1, X_2 \in \mathcal{B}$, then $X \in \mathcal{B}$,

b*) $\mathcal{B}$ is **weakly extension coclosed**: if $0 \to X_1 \to X \to X_2 \to 0$

$\mathcal{B}$, then $X \notin \mathcal{B}$. 
The family of lattices of the form $\mathcal{BIC}(Q)$ properly contains the lattices isomorphic to the weak order on $\mathfrak{S}_n$. 
Biclosed subcategories

Theorem (G.– McConville)

Let $\mathcal{B} = \text{add}(\bigoplus_{i=1}^{k} X_i) \in \mathcal{BIC}(Q)$ and let

$$\pi_{\downarrow}(\mathcal{B}) := \text{add}(\bigoplus_{j=1}^{\ell} X_{ij} : X_{ij} \rightarrow Y \implies Y \in \mathcal{B}).$$
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Then $\pi_\downarrow : \text{BIC}(Q) \rightarrow \text{tors}(\Lambda)$ is a lattice quotient map.
Theorem (G.–McConville)

The lattice $\mathcal{BIC}(Q)$ is semidistributive, congruence-uniform, and polygonal.
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The lattice $\mathcal{BIC}(Q)$ is semidistributive, congruence-uniform, and polygonal. Thus so is $\overrightarrow{EG}(\hat{Q}) \cong \text{tors}(\Lambda)$.

Theorem (Caspard–Le Conte de Poly-Barbut–Morvan 2004, Reading (proves polygonality) in forthcoming book)

The weak order on the symmetric group (in fact, on any finite Coxeter group) is semidistributive, congruence-uniform, and polygonal.
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Some congruence-uniform lattices
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![Diagram showing maximal green sequences connected by polygonal flips]
Theorem (G.–McConville)

If $Q$ is mutation-equivalent to $1 \to 2 \to \cdots \to n$, $\overrightarrow{EG}(\hat{Q})$ is a polygonal lattice whose \textbf{polygons} are of the form
Application: maximal green sequences

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![Diagram of polygons]

Corollary (Conjectured by Brüstle-Dupont-Pérotin for any quiver $Q$)

If $Q$ is mutation-equivalent to $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$, the set of lengths of the maximal green sequences of $Q$ is of the form

$\{\ell_{\min}, \ell_{\min} + 1, \ldots, \ell_{\max} - 1, \ell_{\max}\}$ where

$\ell_{\min} := \text{length of the shortest maximal green sequence of } Q,$

$\ell_{\max} := \text{length of the longest maximal green sequence of } Q.$
Thanks!