Symmeric self-adjoint Hopf categories and a categorical Heisenberg double

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Abstract

We define what we call a *symmetric self-adjoint Hopf* structure on a semisimple abelian category, which is an analog of Zelevinsky's positive self-adjoint Hopf algebra structure for categories. As examples we exhibit this structure on the categories of polynomial functors and equivariant polynomial functors and obtain a categorical manifestation of Zelevinsky's decomposition theorem involving them. It follows from the work of Zelevinsky that every positive self-adjoint Hopf algebra A admits a Fock space action of the Heisenberg double (A, A). We show that the notion of symmetric self-adjoint Hopf category leads naturally to the definition of a categorical analog of such an action and that every symmetric self-adjoint Hopf category admits such an action.

Contents

1 Introduction

In this article we define a categorical analog of a certain class of Hopf algebras: namely those endowed additionally with an inner product with respect to which the maps of multiplication and comultiplication are adjoint. The main idea is that the requirement of adjointness allows us to essentially only consider the structure of multiplication (or dually only the structure of comultiplication) in the algebra or the category.

To give a Hopf structure in this situation amounts to placing certain requirements on the multiplication. This approach will allow us to give a concise definition on the category level, without resorting to the use of complicated diagrams required to express the relationship between the multiplication and comultiplication in the general situation.

Examples of Hopf algebras with adjoint maps of multiplication and comultiplication arise naturally in representation theory. A principal example which we consider throughout the article is the Hopf algebra $\Lambda := \bigoplus_n K(\mathrm{Rep}(S_n))$ with multiplication and comultiplication given by induction and restriction functors and inner product given by the dimensions of the hom-spaces. Certain representation theoretic information, e.g. decomposition into irreducibles, can be derived from studying such algebras. An abstract theory of such algebras was developed by Zelevinsky in the book [Zel81] under the name of positive self-adjoint Hopf (PSH) algebras:

Definition 1.1. A positive self-adjoint Hopf (PSH) algebra is a graded connected Hopf algebra over $\mathbb{Z}$ with an inner product and a distinguished finite orthogonal $\mathbb{Z}$ basis in each grade s.t. multiplication and comultiplication are adjoint and take elements with positive coefficients in the basis to elements with positive coefficients in the basis (that is to say, they are *positive* maps).

We want to analyze this definition and restate it in a form which can be adapted to the category setting.

Given a PSH algebra A we have an adjoint pair of multiplication and comultiplication maps which we will denote m and Δ:

$$A \otimes A \xrightarrow{m} A \qquad\qquad A \xrightarrow{\Delta} A \otimes A$$

The Hopf axiom is the requirement that

$$\forall x, y \in A : \Delta(xy) = \Delta(x)\Delta(y)$$

This can be restated as the commutativity of the following square:

$$\begin{tikzcd} A^{\otimes 4} \arrow[r, "\overline{m}"] & A^{\otimes 2} \\ A^{\otimes 2} \arrow[u, "\Delta^{\otimes 2}"] \arrow[r, "m"] & A \arrow[u, "\Delta"'] \end{tikzcd}$$

where

$$\overline{m}(x \otimes y \otimes z \otimes w) = m(x \otimes z) \otimes m(y \otimes w)$$
$$\Delta^{\otimes 2}(x \otimes y) = \Delta(x) \otimes \Delta(y)$$

Using the adjointness of m and Δ this square can be obtained from the square

$$\begin{tikzcd} A^{\otimes 4} \arrow[r, "\overline{m}"] \arrow[d, "m^{\otimes 2}"'] & A^{\otimes 2} \arrow[d, "m"] \\ A^{\otimes 2} \arrow[r, "m"] & A \end{tikzcd} \tag{1.1}$$

by replacing the verticals with their adjoints. This operation is called "taking the mate" of the square. We will say that a square satisfies the *Beck-Chevalley condition* if its mate commutes.

Observe that the square (1.1) corresponds to the Cartesian square of finite sets

$$\begin{tikzcd} A^{\otimes 4} \arrow[r, "\overline{m}"] \arrow[d, "m^{\otimes 2}"'] & A^{\otimes 2} \arrow[d, "m"] \\ A^{\otimes 2} \arrow[r, "m"] & A \end{tikzcd} \qquad \leftarrow \qquad \begin{tikzcd} {[4]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & {[2]} \arrow[d] \\ {[2]} \arrow[r] & {[1]} \end{tikzcd}$$

where we denote the finite set $\{1, 2, \ldots, n\}$ by $[n]$. More generally, given the multiplication map $m : A \otimes A \to A$, we have for every map of finite sets $a : S \to T$ an obvious extension $m_a : A^{\otimes S} \to A^{\otimes T}$. The Hopf axiom implies that all squares in the algebra formed using the above extension, that correspond to Cartesian squares of finite sets, satisfy the Beck-Chevalley condition.

These observations allow us to reformulate the Definition 1.1 as follows: Denote by $\mathbb{Z}\text{-}\operatorname{Mod}^{gr}_{ad}$ the category of graded free $\mathbb{Z}$ modules with chosen finite basis in each grade and positive graded maps which admit positive graded adjoints (with respect to the inner product induced by the chosen basis). Then the following holds:

Theorem 1. *A PSH algebra is the same as a functor*

$$\mathcal{A} : \mathbf{FinSet} \to \mathbb{Z}\text{-}\operatorname{Mod}^{gr}_{ad}$$

which takes disjoint union to tensor product (i.e. a symmetric monoidal functor), and sends Cartesian squares to squares satisfying the Beck-Chevalley condition.

Remark 1.2. The requirement of grading is not very essential in the definition, since from Zelevinsky's decomposition theorem for PSH algebras ([Zel81] §2) it can be seen that the grading is determined freely by setting its values on irreducible primitive elements. Therefore in the categorical setting we will omit the requirement of having a grading (see also Remark 2.17).

The proof of Theorem 1 is straightforward. In particular, we note that the unit and counit maps are given by the image of the map $\emptyset \to [1]$ and the connectedness follows from the fact that the image of the Cartesian square

$$\begin{tikzcd} \emptyset \arrow[r, equal] \arrow[d, equal] & \emptyset \arrow[d] \\ \emptyset \arrow[r] & {[1]} \end{tikzcd}$$

satisfies the Beck-Chevalley condition.

Given in this form, the definition of a PSH algebra can be categorified to define a class of objects which we call the *symmetric self-adjoint Hopf categories.* We replace the functor $\mathcal{A} : \mathbf{FinSet} \to \mathbb{Z}\text{-}\operatorname{Mod}^{gr}_{ad}$ with a functor $\mathcal{H} : \mathbf{FinSet} \to \mathbf{Cat}_{ex}$, where $\mathbf{Cat}_{ex}$ is the category of semisimple abelian $\Bbbk$-linear categories with objects of finite length and 1-morphisms exact functors. The natural analog of the tensor of $\mathbb{Z}$-modules is the Deligne tensor of categories (see Definition 2.9). Considering $\mathbf{Cat}_{ex}$ as an $(\infty, 2)$-category, one can show that Deligne tensor product gives a symmetric monoidal structure on it and work with this structure following the approach outlined by Lurie in the book [Lur11]. The 2-categorical Beck-Chevalley condition is described in Appendix B, and we prove in §2.7 that in $\mathbf{Cat}_{ex}$ it is equivalent to the square being comma - a 2-categorical generalization of a Cartesian square. We can thus formulate the following 2-categorical analog of the statement from the Theorem 1:

Definition 1.3. A *symmetric self-adjoint Hopf (SSH)* category is a symmetric monoidal functor $\mathbf{FinSet} \xrightarrow{\mathcal{H}} \mathbf{Cat}_{ex}$, considered as $(\infty, 2)$-categories, which preserves *comma squares*.

Given such a functor $\mathcal{H}$ with $\mathcal{H}([1]) \cong \mathcal{C}$ we will say that $\mathcal{H}$ gives a structure of SSH category on $\mathcal{C}$ and that $\mathcal{C}$ is an SSH category. The associated technical details, including the precise description of the data needed to define a symmetric monoidal functor in the ∞-categorical setting are discussed in §2. One upshot is that giving a symmetric monoidal functor $\mathcal{H}$ means giving quite a rich structure of 1-and 2-morphisms in $\mathbf{Cat}_{ex}$. In particular, an SSH category $\mathcal{C}$ has a canonically defined symmetric monoidal structure. The Grothendieck group of $\mathcal{C}$ has a canonical structure of PSH algebra, as shown in §2.5. Formulating the condition on $\mathcal{H}$ with comma squares instead of Beck-Chevalley condition is important for defining the categorical Fock space representation of the Heisenberg double for an SSH category - more on this subject below, and in §5.

The principal examples of PSH algebras considered in the book [Zel81] are the algebras $\Lambda := \bigoplus_n K(\mathrm{Rep}(S_n))$ and $\Lambda_G := K\left(\bigoplus_n \mathrm{Rep}(S_n[G])\right)$ where $S_n[G]$ is the wreath product $G^n \rtimes S_n$. The parallel examples of SSH categories that we consider in this work are the category $\mathcal{P}$ of polynomial functors defined by Friedlander and Suslin in [FS97] and the category $\mathcal{P}_G$ of equivariant polynomial functors defined in §6.1. In characteristic 0, both of these categories are semisimple and the SSH structure descends to the PSH structure on $\Lambda \cong K(\mathcal{P})$ and $\Lambda_G \cong K(\mathcal{P}_G)$, respectively. These examples are treated in §3 and §6.

Remark 1.4. The setting of semisimple abelian categories considered in this article is rather restrictive and doesn't include the important examples which should naturally exhibit an SSH strucutre, such as the category of polynomial functors over a field of positive characteristic. We believe that the correct generalization for this setting is that of stable ∞-categories. Since the examples we consider in this article are semisimple we prefer to delay the more general treatment to a separate work.

1.1 The categorical Heisenberg double and the Fock space action

As a first application of the concept of SSH category we prove that every SSH category admits a categorical analog of a Heisenberg double Fock space action.

In Proposition 4.4 we show (following Zelevinsky) that any positive self-adjoint Hopf algebra A admits a canonically defined action of the Heisenberg double (A, A).This action is given by multiplication by the elements of A and their adjoints which satify certain relations making it a Heisenberg double action. Using the SSH structure on a category we can attempt to directly categorify the proof of Proposition 4.4 by replacing these relations with canonically constructed 2-isomorphisms (5.1) in §5.

The nice feature of these 2-isomorphisms is that they arise as a consequence of certain squares - that are part of the SSH structure - satisfying the Beck-Chevalley condition, which in $\mathbf{Cat}_{ex}$ is the same as being comma. In fact, we notice that these squares are images of comma squares in the category $\mathrm{Heis}(\mathcal{H})$, defined in §5.3, via a canonical projection $\mathrm{Heis}(\mathcal{H}) \to \mathbf{Cat}_{ex}$. From the definition of $\mathrm{Heis}(\mathcal{H})$, this projection preserves comma squares because $\mathcal{H}$ does. We call the category $\mathrm{Heis}(\mathcal{H})$ equipped with this canonical projection the *categorical Heisenberg double* of an SSH functor $\mathcal{H}$. This gives a more natural definition for categorification of the Heisenberg action that isn't just directly replacing the relations in the algebra by the isomorphisms (5.1) (even though it, of course, in particular implies the existence of these). For every comma square in Heis we get an isomorphism in $\mathbf{Cat}_{ex}$ via the Beck-Chevalley condition.

Interestingly, some important comma squares in $\mathrm{Heis}(\mathcal{H})$ which have an explicit natural construction (see §5.3) contain *non-invertible* 2-morphisms. This means that such squares are a purely categorical construct which is not detectable on the algebra level, where they descend to non-commutative diagrams.

1.2 Categorification of the Heisenberg algebra

The Heisenberg double associated to the algebra Λ of symmetric functions is also called the Heisenberg algebra of infinite rank. This algebra is usually defined in the literature in terms of an infinite number of generators and relations; there are several common presentations, some of which, as well as their relation to the generator-independent definition used in this article are outlined in §4.2. Khovanov proposed, in [Kho10], an approach to categorification of the Fock space action of this algebra in a diagrammatic language of generators and relations for endofunctors of a category and 2-morphisms between them. In [HY13] Hong and Yacobi describe this action on the category $\mathcal{P}$.

The original motivation for this article was our desire to understand which parts of Khovanov's construction can be derived from the categorical version of a PSH structure. It seems that an SSH structure implies the existence of the Heisenberg action in the sense of Khovanov under certain additional explicit conditions on the adjunction data (which hold in $\mathcal{P}$, for example). As a consequence this should give a way to construct Khovanov's Heisenberg action for any SSH category and a categorical analog of "primitive" element of the PSH algebra.

In this article we treat the case of $\mathcal{P}$ as an example of our general construction of categorical Heisenberg double action, thus getting a different approach to categorification of Fock space action of infinite Heisenberg algebra. The constructions in the case of $\mathcal{P}$ are spelled out in §5.4. In the recent work [SY13] by Savage and Yacobi the authors propose to consider the isomorphisms (5.1) for categorifying the Fock space representation of the Heisenberg doubles given by the K-groups of categories of modules over towers of algebras and construct these for a certain class of examples. It would be interesting to attempt to formulate their results in (some generalization of) the SSH categories framework, and this might provide additional insight into the structure inherent in these examples.

1.3 Zelevinsky's decomposition theorem

Zelevinsky's main theorem about PSH algebras is that any such algebra is isomorphic to the tensor products of many copies of the PSH algebra Λ (see [Zel81] §2 for the precise formulation). His proof is somewhat combinatorial in nature, and gives the morphism to the tensor product only up to a non-canonical choice. In section 6 we construct a canonical equivalence of SSH categories $\mathcal{P}_G$ and $\mathcal{P}^{\otimes \operatorname{Irr} G}$ which descends to the isomorphism from Zelevinsky's theorem. We conjecture that a similar decomposition statement is true for any SSH category. The formulation of the precise statement involves a more detailed analysis of the analogue in the categorical setting for *primitive elements* - a notion introduced in [Zel81] which is a main tool in proving the decomposition theorem. This is a subject which we plan to explore in a separate work.

1.4 The braided monoidal case

As we noted earlier, an SSH structure naturally defines a symmetric monoidal structure on the category. The reason for it is our choice of **FinSet** as the indexing category. Modifying the definition to get braided monoidal structure on the underlying category would produce a framework for dealing with interesting examples such as categories of representations of quantum groups, or of $GL(n, \mathbb{F}_q)$, and categorical objects connected to Hall algebras.

1.5 Acknowledgments

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1.6 Notations

- $\Bbbk$ - a field of characteristic 0
- **Vect** - the category of finite dimensional vector spaces over $\Bbbk$
- $[n]$ - the set $\{1, 2, \ldots, n\}$
- $[n]_*$ - the set $\{1, 2, \ldots, n, *\}$ in the category of pointed finite sets $\mathbf{FinSet}_*$
- $\mathrm{Sh}(S)$ - the category of sheaves of finite dimensional vector spaces on S
- identity morphisms will be drawn as $=\!=\!=$ in diagrams
- we will refer to a square of the form $\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "\swarrow\alpha"] \\ C \arrow[r] & D \end{tikzcd}$ as α

2 Symmetric self-adjoint Hopf categories

2.1 Remarks about the framework

In this section we give an analog of the PSH algebra structure on an abelian semisimple $\Bbbk$-linear symmetric monoidal category $\mathcal{C}$. We consider functors of multiplication and comultiplication between arbitrary powers of $\mathcal{C}$ and a coherent system of natural transformations between these functors lifting the relations in the algebra, such as the Hopf axiom. We want the functors of multiplication and comultiplication to be adjoint, and in the interest of simplicity we take them to be exact functors between semisimple abelian categories.

This naturally leads us to consider a symmetric monoidal functor $\mathcal{H}$ from the category **FinSet** of finite sets with product given by disjoint union to the category of $\Bbbk$-linear semisimple abelian categories with product given by the Deligne tensor of categories. To work properly with the the symmetric monoidal structure on the latter category, one should work in the framework of $(\infty, 2)$-categories. That is, we define $\mathcal{H}$ to be a functor between $(\infty, 2)$-categories. We give the definition of the functor $\mathcal{H}$ following the approach of Lurie in [Lur11] and refer to this book as a source for all the results we need. We do so despite the fact that everything in the book [Lur11] is formulated for $(\infty, 1)$-categories, since the results we cite should work in any reasonable model for $(\infty, 2)$-categories as well, but unfortunately as of now we know of no similarly comprehensive published source.

We have been told that this topic will be adressed by Gaitsgory and Rozenblyum in their upcoming book. In an overview given by Nick Rozenblyum in Caesarea the $(\infty, 2)$-categories were considered as objects in the category $\mathrm{Psh}(\Delta^2, S)$ of presheaves with values in the category of spaces $S = \mathrm{Psh}(\Delta, \mathbf{Set})$). However, any model of choice in which the analogs of the results from [Lur11] which we cite can be proven is suitable for formulating and working with our definitions.

The model used notwithstanding, if $\mathcal{C}$ is an ordinary 1- or 2- category, it can be considered as an $(\infty, 2)$-category in a concrete way. We sometimes denote a corresponding ∞ version by $N(\mathcal{C})$ or just $\mathcal{C}$ by abuse of notation when there is no ambiguity. Likewise, when we state that an $(\infty, 2)$-category $\mathcal{C}$ is a 1- or 2-category, we mean that it is 2- or 3- coskeletal respectively.

2.2 Symmetric monoidal functors

By a symmetric monoidal functor we mean here an ∞ symmetric monoidal functor in the sense of Lurie (see [Lur11]).

That is, we consider a functor between the coCartesian fibrations (in the $(\infty, 2)$-category setting) $\mathbf{FinSet}^{\sqcup} \to N(\mathbf{FinSet}_*)$ and $\mathbf{Cat}_{ex}^{\otimes} \to N(\mathbf{FinSet}_*)$ defined as follows (cf. [Lur11] §2.4.3.1 and §6.3.1)

Definition 2.1 ($\mathbf{FinSet}^{\sqcup}$)**.**

- An object of $\mathbf{FinSet}^{\sqcup}$ is a collection $(S_*, S_1, \ldots, S_n)$ of finite sets, with S_* being a terminal object in **FinSet**, i.e. a one point set.
- A map $(S_*, S_1, \ldots, S_n) \to (T_*, T_1, \ldots, T_m)$ is a map $[n]_* \xrightarrow{\varphi} [m]_*$ in $\mathbf{FinSet}_*$ and a collection of functions of finite sets $S_i \to T_{\varphi(i)}$ for all i.

Note. Our notation is different from the one in [Lur11] in that there the sets S_* are omitted. We will mostly omit S_* from the notation as well.

Definition 2.2. $\mathbf{Cat}_{ex}$ is the category of semisimple abelian $\Bbbk$-linear categories whose objects are of finite length. The 1-morphisms of $\mathbf{Cat}_{ex}$ are exact functors.

Remark 2.3. The category $\mathbf{Cat}_{ex}$ has a canonical embedding into the category $\mathbf{Cat}_{\infty}^{\mathrm{Ex}}$ of stable ∞-categories defined in [Lur11]. Our definition of SSH categories can be generalized to this setting. See also Remarks 1.4 and 5.4.

Definition 2.4 ($\mathbf{Cat}_{ex}^{\otimes}$)**.**

- An object of $\mathbf{Cat}_{ex}^{\otimes}$ is a collection $(\mathcal{C}_*, \mathcal{C}_1, \ldots, \mathcal{C}_n)$ of abelian categories, with $\mathcal{C}_*$ being the final object in $\mathbf{Cat}_{ex}$, i.e. a one object trivial category.
- A map $(\mathcal{C}_*, \mathcal{C}_1, \ldots, \mathcal{C}_n) \to (\mathcal{D}_*, \mathcal{D}_1, \ldots, \mathcal{D}_m)$ is a map $[n]_* \xrightarrow{\varphi} [m]_*$ in $\mathbf{FinSet}_*$ together with a collection of functors

  $$f_j : \prod_{\varphi(i)=j} \mathcal{C}_i \to \mathcal{D}_j$$

  exact in every variable.

  Note. If $\varphi^{-1}(j) = \emptyset$, then we should give a map from the empty product, which is the one object category, to $\mathcal{D}_j$, which is the same as specifying an element in $\mathcal{D}_j$. There is no requirement of exactness since there are no variables.

- a 2-cell in $\mathbf{Cat}_{ex}^{\otimes}$ between two maps $F, G : (C_i)_{i \in I_*} \to (D_j)_{j \in J_*}$ over $\varphi : I_* \to J_*$ is a collection of 2-morphisms in $\mathbf{Cat}_{ex}$ between the functors defining F, G, i.e. natural transformations $\varphi_j : F_j \to G_j$.

  Note. When $\varphi^{-1}(j) = \emptyset$, then giving $\alpha_j : F_j \to G_j$ amounts to specifying a morphism between the specified objects.

Definition 2.5. A symmetric monoidal functor is a functor of coCartesian fibrations over $N(\mathbf{FinSet}_*)$, which sends coCartesian lifts to coCartesian lifts.

Finally, our main definition is:

Definition 2.6. A *symmetric self-adjoint Hopf (SSH)* category is a symmetric monoidal functor $\mathbf{FinSet}^{\sqcup} \xrightarrow{\mathcal{H}} \mathbf{Cat}_{ex}^{\otimes}$ which preserves *comma squares* in the fiber over $[1]_*$.

Notation 2.7. Given such a functor $\mathcal{H}$ with $\mathcal{H}([1]) \cong \mathcal{C}$ we will say that $\mathcal{H}$ gives a structure of SSH category on $\mathcal{C}$ and that $\mathcal{C}$ is an SSH category.

In a 1-category, e.g. **FinSet**, a comma square is the same as a Cartesian square. We discuss comma squares in $\mathbf{Cat}_{ex}$ and $\mathbf{FinSet}^{\sqcup}$ in §2.7.

2.3 CoCartesian lifts

The definition of coCartesian arrow is given in [Lur09], but when working with a coCartesian fibration, we have the following simpler description (see [Lur09] Proposition 2.4.2.8):

Lemma 2.8. *An arrow $x \to y$ is a coCartesian lift over $[n]_* \xrightarrow{\varphi} [m]_*$ iff it is initial in the category of arrows over φ with source x. (This is what is called a* locally *coCartesian lift in [Lur09]).*

Let us analyze what this amounts to in our two fibrations, where for simplicity we take $m = [1]_*$ and φ the (unique) active map.

In $\mathbf{FinSet}^{\sqcup}$ this is a collection of maps $S_i \to T$ which is initial among collections of maps $S_i \to T'$, i.e. a presentation of T as the disjoint union (or coproduct) of the S_i.

In $\mathbf{Cat}_{ex}^{\otimes}$ this is a functor $\prod \mathcal{C}_i \to \mathcal{D}$, exact in every variable, and initial among such functors. This is precisely what it means to present $\mathcal{D}$ as the Deligne tensor of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ (see [Del90]).

Definition 2.9. Let A, B be abelian categories, and $A \times B \xrightarrow{\alpha} C$ a functor which is right exact in each variable. We say that α presents C as $A \otimes B$ if, for any abelian D, precompostion with α gives an equivalence of categories

$$\mathrm{Fun}_{\text{right exact}}(C, D) \xrightarrow{\sim} \mathrm{Fun}_{\text{r.ex. in each vairable}}(A \times B, D)$$

Remark 2.10. This definition can be rephrased as saying that the bi-right exact map $A \times B \xrightarrow{\alpha} C$ is *initial* in the 2-category of all bi-right exact maps $A \times B \xrightarrow{\alpha} C'$.

For our purposes we note that in $\mathbf{Cat}_{ex}$ the Deligne tensor always exists, and is semisimple (see e.g. [LF13]).

In light of this, we will sometimes write $\boxtimes : (\mathcal{C}_i) \to \mathcal{D}$ to denote coCartesian arrows in $\mathbf{Cat}_{ex}^{\otimes}$, and $+ : (S_i) \to T$ to denote coCartesian arrows in $\mathbf{FinSet}^{\sqcup}$.

2.4 Unravelling the definition

In §2.7 we will prove

Proposition 2.11. *In* $\mathbf{Cat}_{ex}$ *a square is comma iff it satisfies the Beck-Chevalley condition.*

In view of this, the data we need to give the SSH category structure is the following:

- For every finite set S, a category $\mathcal{H}(S)$.
- For every finite collection of maps of finite sets $(S_i \xrightarrow{\varphi_i} T)_{i\in I}$, a functor $\mathcal{H}(\varphi_i) : \prod_i \mathcal{H}(S_i) \to \mathcal{H}(T)$.
- For every commutative square

$$\begin{tikzcd}
(S_i)_{i\in I} \arrow[r] \arrow[d] & (Q_k)_{k\in K} \arrow[d] \\
(T_j)_{j\in J} \arrow[r] & U
\end{tikzcd}$$

(where commutativity is understood separately for each i), a 2-commutative square

$$\begin{tikzcd}
\mathcal{H}((S_i)_{i\in I}) \arrow[r] \arrow[d] & \mathcal{H}((Q_k)_{k\in K}) \arrow[d] \arrow[dl, Rightarrow, "\sim"'] \\
\mathcal{H}((T_j)_{j\in J}) \arrow[r] & \mathcal{H}(U)
\end{tikzcd}$$

And it has to satify the following properties:

- 3-cells in $\mathbf{FinSet}^{\sqcup}$ map to 3-cells in $\mathbf{Cat}_{ex}^{\otimes}$, e.g. commutative cubes go to commutative cubes.
- CoCartesian arrows map to coCartesian arrows.
- Comma squares of sets in the fiber over $[1]_*$ map to squares of categories satisfying the Beck-Chevalley condition.

Notation 2.12. The restriction of the functor $\mathcal{H}$ to the fiber over $[1]_*$ gives a functor that we will denote $\mathcal{H}^1 : \mathbf{FinSet} \to \mathbf{Cat}_{ex}$ which takes comma squares to comma squares. One can think about the images of the maps in $\mathbf{FinSet}$ under $\mathcal{H}^1$ and their adjoints as the categorical analogs of the maps of multiplication and comultiplication in the Hopf algebra (for any number of variables). Hence we will denote the image of the arrows $S \xrightarrow{a} T$ by m_a and the image of the arrow $[2] \to [1]$ will be denoted just by m.

Let $\mathcal{C} = \mathcal{H}([1])$ (in other words $\mathcal{H}$ gives an SSH structure on the category $\mathcal{C}$). Then elsewhere in the article we will denote the image of a finite set U under $\mathcal{H}$ by $\mathcal{C}^{\otimes U}$. This notation is motivated by the fact that, as we noted earlier, the image of U should satisfy the universal property of being a Deligne tensor.

2.5 Basic properties

Throughout, let $\mathcal{H}$ be a SSH structure on $\mathcal{C}$.

Proposition 2.13. *$\mathcal{H}(\emptyset)$ is canonically equivalent to* **Vect**.

Proof. Let $\alpha : [0]_* \to [1]_*$ be the unique map. In $\mathbf{Cat}_{ex}^{\otimes}$, over $[0]_*$, there is only the list (C_*) consisting of the trivial category. Therefore, a map in $\mathbf{Cat}_{ex}^{\otimes}$ over α is a map $(C_*) \to (C_*, D)$, which amounts to a functor from the empty product to D, i.e. the specification of an object in D.

We have:

1. The arrow $(S_*) \to (S_*, \emptyset)$ in $\mathbf{FinSet}^{\sqcup}$ is a coCartesian arrow over α.
2. The arrow $(C_*) \to (C_*, \mathbf{Vect})$ in $\mathbf{Cat}_{ex}^{\otimes}$ which sends the empty product to $\Bbbk$, is a coCartesian arrow over α.

The first is obvious, and second is just saying that a choice of an element in D is *essentially* the same as a functor $\mathbf{Vect} \to D$.

As a result, the first arrow must go to an arrow which is canonically equivalent to the second arrow (canonically because coCartesian is a universal property, as noted in 2.8), and in particular $\mathcal{H}(\emptyset)$ is canonically equivalent to **Vect**. □

Proposition 2.14. *$\mathcal{H}$ defines a symmetric monoidal structure on $\mathcal{C}$.*

Proof. Consider the section s of $\mathbf{FinSet}^{\sqcup} \to \mathbf{FinSet}_*$ given by $[n]_* \mapsto ([1], \dots, [1])$ which sends a map $\alpha : [m]_* \to [n]_*$ to the appropriate collection of identity maps (note also that there is such a section for any finite set S in place of $[1]$ - this is related to the fact that a symmetric monoidal functor $\mathbf{FinSet}^{\sqcup} \to \mathcal{C}^{\otimes}$ factors through $\mathrm{CAlg}(\mathcal{C})^{\otimes}$ - see [Lur11] 3.2.4.9).

Composing this section with the map $\mathbf{FinSet}^{\sqcup} \to \mathbf{Cat}_{ex}^{\otimes}$ we get a section $\bar{s}$ of $\mathbf{Cat}_{ex}^{\otimes} \to \mathbf{FinSet}_*$, which gives a symmetric monoidal structure on $\bar{s}([1]) = H([1]) = \mathcal{C}$. □

Explicitly, this monoidal structure is given by applying $\mathcal{H}$ to the map $([1], [1]) \to ([1])$ given by the two copies of $\mathrm{Id} : [1] \to [1]$ which lies over the active map $[2]_* \to [1]_*$.

Let us denote this map by $(F, G) \mapsto F \otimes G$, where F, G are objects in $\mathcal{C}$

Remark 2.15. The map $(F, G) \mapsto F \otimes G$ factors essentially uniquely as $\mathcal{H}$ applied to the composition

$$\begin{tikzcd} {[1]} \arrow[dr] & & \\ & {[2]} \arrow[r] & {[1]} \\ {[1]} \arrow[ur] & & \end{tikzcd}$$

The factorization is given on $F, G \in \mathcal{C}$ as $(F, G) \mapsto F \boxtimes G \mapsto m(F \boxtimes G)$ i.e. a coCartesian arrow followed by an arrow over $\mathrm{Id}_{[1]_*}$.

Corollary 2.16. *The functor $\mathcal{H}$ canonically defines all the features of a PSH structure on the Grothendieck group $K(\mathcal{C})$, except a grading.*

Proof. Let $A := K(\mathcal{C})$.

Since all functors involved are exact, the functor $\mathbf{FinSet}^{\sqcup} \to \mathbf{Cat}_{ex}^{\otimes}$ descends (by taking K-groups) to a symmetric monoidal functor of ordinary categories $\mathbf{FinSet}^{\sqcup} \to \mathbb{Z}\text{-}\operatorname{Mod}_{ad}^{\otimes}$, where $\mathbb{Z}\text{-}\operatorname{Mod}_{ad}^{\otimes}$ is the coCartesian fibration corresponding to the symmetric monoidal category of free $\mathbb{Z}$-modules with a chosen basis, and positive maps which admit positive adjoints (with respect to the inner product induced by the chosen basis).

The fact that all morphisms are positive is obvious, since they come from exact functors between semisimple categories.

The section which gives the monoidal structure on $\mathcal{C}$ descends to a section which gives the structure of algebra on A. The Hopf axiom comes from the continuity of the original functor, as follows:

Consider the comma square in $\mathbf{FinSet}^{\sqcup}$, and its image under $\mathcal{H}$

$$\begin{tikzcd} {[4]} \arrow[r, "a"] \arrow[d, "b"] & {[2]} \arrow[d, "c"] \\ {[2]} \arrow[r, "d"] & {[1]} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathcal{H}([4]) \arrow[r, "\mathcal{H}(a)"] \arrow[d, "\mathcal{H}(b)"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{H}([2]) \arrow[d, "\mathcal{H}(c)"] \\ \mathcal{H}([2]) \arrow[r, "\mathcal{H}(d)"] & \mathcal{H}([1]) \end{tikzcd}$$

Since the square is comma, its image is required to be comma, and hence satisfies the Beck-Chevalley condition, as noted in 2.11. Taking K groups, we get the commutative square of algebras

$$\begin{tikzcd} A^{\otimes 4} \arrow[r, "m_{12} \otimes m_{34}"] \arrow[d, "m_{13} \otimes m_{24}"'] & A^{\otimes 2} \arrow[d, "m"] \\ A^{\otimes 2} \arrow[r, "m"] & A \end{tikzcd}$$

The Hopf axiom amounts to showing that this square with the top and bottom arrows replaced by adjoints should commute. But replacing top and bottom with adjoints in the square of algebras is the same as taking the left mate of the square α and then taking K groups. As we mentioned, the square α satisfies the Beck-Chevalley condition, hence it's left mate has an isomorphism in the middle, so we conclude that the square of algebras commutes.

Finally, the connectedness of $K(\mathcal{C})$ follows from the discussion in §2.6.1. $\square$

Remark 2.17 (Grading). In the definition of a PSH algebra A, it is required that A is graded, and that every grade have a given finite orthonormal basis. The analog of this in the categorical setting would be to require that the images of objects under $\mathcal{H}$ are graded and have a finite number of irreducibles in every grade. In the examples we consider in this paper, it is always the case that such a grading exists, but we don't need to use it in any of our constructions or proofs, and it seems natural to omit this requirement from our general definition of an SSH category (see also Remark 1.2).

Proposition 2.18. *For any finite set T_0 we have a natural SSH structure on $\mathcal{H}(T_0)$.*

Proof. Define a functor by $S \mapsto \mathcal{H}(S \times T_0)$, then it is obviously an SSH functor, and it sends $[1]$ to $\mathcal{H}(T_0)$. $\square$

2.6 The categorical analogs of Hopf algebra structures

2.6.1 Connectedness

Recall that a PSH algebra A is defined to be a *connected* Hopf algebra, namely it has unit and counit morphisms $\mathbb{Z} \to A_0$ and $A \to \mathbb{Z}$ such that they give an isomorphism of A_0 with $\mathbb{Z}$.

This is categorified as follows: The unit and counit in $K(\mathcal{C})$ are the image of the map $\emptyset \to 1$ in $\mathbf{FinSet}^{\sqcup}$ and its left adjoint. Denote the image of this map under $\mathcal{H}$ by $m_\emptyset$ and its adjoint by $\Delta_\emptyset$.

Consider the diagrams

$$\begin{tikzcd} \emptyset \arrow[r, equal] \arrow[d, equal] & \emptyset \arrow[d] \\ \emptyset \arrow[r] & 1 \end{tikzcd} \qquad\qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & 1 \arrow[d, equal] \\ 1 \arrow[r, equal] & 1 \end{tikzcd}$$

They go to squares in $\mathbf{Cat}_{ex}$

$$\begin{tikzcd} \mathbf{Vect} \arrow[r, equal] \arrow[d, equal] & \mathbf{Vect} \arrow[d, "m_\emptyset"] \arrow[dl, Rightarrow, "\mathrm{Id}"'] \\ \mathbf{Vect} \arrow[r, "m_\emptyset"'] & \mathcal{C} \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbf{Vect} \arrow[r, "m_\emptyset"] \arrow[d, "m_\emptyset"'] & \mathcal{C} \arrow[d, equal] \arrow[dl, Rightarrow, "\mathrm{Id}"'] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

their left mates are

$$
\begin{tikzcd}
\mathbf{Vect} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Nwarrow\alpha"] & \mathbf{Vect} \arrow[d, "m_\emptyset"] \\
\mathbf{Vect} & \mathcal{C} \arrow[l, "\Delta_\emptyset"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{Vect} \arrow[d, "m_\emptyset"'] \arrow[dr, phantom, "\Nwarrow\beta"] & \mathcal{C} \arrow[l, "\Delta_\emptyset"'] \arrow[d, equal] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
$$

The left square came from a comma square, so contains an isomorphism, i.e. α is invertible, but β need not be, and in a non trivial situation will not be.

Define $\mathcal{C}_0$ to be the full subcategory of objects $X \in \mathcal{C}$ for which $\beta : X \to m_\emptyset \Delta_\emptyset X$ is invertible.

Proposition 2.19. *$m_\emptyset, \Delta_\emptyset$ give an adjoint equivalence between* **Vect** *and $\mathcal{C}_0$.*

Proof. All we need to show is that $m_\emptyset$ always lands in $\mathcal{C}_0$. This holds because α, β are just the counit and unit of the adjunction $(m_\emptyset \dashv \Delta_\emptyset)$, so the composition

$$m_\emptyset V \xrightarrow{\beta} m_\emptyset \Delta_\emptyset m_\emptyset V \xrightarrow{\alpha} m_\emptyset V$$

is the identity of $m_\emptyset V$. In particular β is invertible for $m_\emptyset V$. $\square$

2.6.2 The Hopf axiom

In the notation of 2.12, for any comma square of maps of sets we have the corresponding diagram in $\mathbf{Cat}_{ex}$

$$
\begin{tikzcd}
S \arrow[r, "a"] \arrow[d, "c"] & T \arrow[d, "b"] \\
R \arrow[r, "d"] & U
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathcal{C}^{\otimes S} \arrow[r, "m_a"] \arrow[d, "m_c"] \arrow[dr, phantom, "\overset{\wr}{\Swarrow}"] & \mathcal{C}^{\otimes T} \arrow[d, "m_b"] \\
\mathcal{C}^{\otimes R} \arrow[r, "m_d"] & \mathcal{C}^{\otimes U}
\end{tikzcd}
$$

(Here $S = T \times_U R$). Denoting the left adjoint to m by Δ^l we can consider the left mate of this square

$$
\begin{tikzcd}
\mathcal{C}^{\otimes S} \arrow[d, "m_c"'] \arrow[dr, phantom, "\Nwarrow"] & \mathcal{C}^{\otimes T} \arrow[l, "\Delta^l_a"'] \arrow[d, "m_b"'] \\
\mathcal{C}^{\otimes R} & \mathcal{C}^{\otimes U} \arrow[l, "\Delta^l_d"']
\end{tikzcd}
$$

And the Beck-Chevalley condition tells us that the 2-morphism in this square is invertible. This can be viewed as a natural categorification of the Hopf axiom in the algebra "for any number of variables".

This system of isomorphisms includes a compatibility with the monoidal structure. If we consider a commutative cube of sets, it goes to a commutative cube in $\mathbf{Cat}_{ex}$, and its left mate is also a commutative cube (see Appendix B), which gives an explicit system of relations between the Hopf isomorphisms and the monoidal structure isomorphisms.

Taking the right adjoint of m gives an additional system of "Hopf isomorphisms".

2.7 Comma squares

A comma square in an $(\infty, 2)$-category can be thought of as a square $\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ C \arrow[r] & D \end{tikzcd}$ which is final in the category of squares which share its right and bottom sides. The precise definition can be formulated in the language of generalized limits (see D.2 for details). In $\mathbf{Cat}_{ex}$ this is equivalent to the usual definition of a comma square in 2-category, namely for any square $\begin{tikzcd} E \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, phantom, "\swarrow"] \\ C \arrow[r] & D \end{tikzcd}$ there exists an "essentially unique" oriented commutative cube

$$\begin{tikzcd}
 & C \arrow[rr] \arrow[dd, equal] & & D \arrow[dd, equal] \\
E \arrow[ur] \arrow[rr] \arrow[dd] & & B \arrow[ur] \arrow[dd, equal] & \\
 & C \arrow[rr] & & D \\
A \arrow[ur] \arrow[rr] & & B \arrow[ur] &
\end{tikzcd} \tag{2.1}$$

where vertices are objects, edges are 1-morphisms and faces are 2-morphisms composed as indicated by the arrows in the diagram (more on the subject of commutative cubes in Appendix A) In **FinSet**, or any 1-category, a comma square is the same as a Cartesian square. We will now give a proof of Proposition 2.11 characterizing comma squares in $\mathbf{Cat}_{ex}$.

Proof of Proposition 2.11. We want to prove that in $\mathbf{Cat}_{ex}$ a square is comma iff it satisfies the Beck-Chevalley condition. For the definitions and details regarding the Beck-Chevalley condition see Appendix B.

Let

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & B \arrow[d, "g"] \arrow[dl, phantom, "\swarrow\alpha"] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

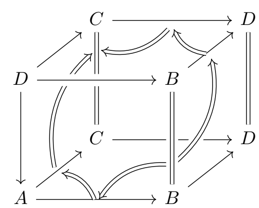
be a square, and suppose it is comma. Denote the 2-morphism in its left mate by α_L. We want to show that α_L is invertible. For any functor $x \in \mathbf{Cat}_{ex}$, denote by x_L, x_R the left and right adjoints of x, and consider the square

$$\begin{tikzcd}
D \arrow[d, "f_L"'] \arrow[dr, phantom, "="] & B \arrow[l, "g"'] \arrow[d, "g"] \\
C & D \arrow[l, "f_L"]
\end{tikzcd}$$

Take its right mate to get the square

$$\begin{tikzcd} D \arrow[r, "g_R"] \arrow[d, "f_L"'] \arrow[dr, phantom, "\swarrow\beta"] & B \arrow[d, "g"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

which has the same right and bottom sides as our original square. As a consequence we have a commutative cube



Taking the left mate of this cube as described in Lemma B.3, we get a commutative cube that on one side has the composition of three identity morphisms and on the other side has a composition of the form $\gamma_1 \circ \gamma_2 \circ \alpha_L$. In particular, α_L is invertible, as required.

In the other direction, suppose that our originial square α satisfies the Beck-Chevalley condition. We want to show it is comma using the definitions in D.2. Let X be the category of squares with bottom right corner $\begin{tikzcd} & B \arrow[d, "g"] \\ C \arrow[r, "f"] & D \end{tikzcd}$ as described in D.2. We need to show that the map $X_{/\alpha} \xrightarrow{p} X$ has weakly contractible fibers.

Consider a square

$$\begin{tikzcd} M \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\swarrow\beta"] & B \arrow[d, "g"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

and let S be the fiber of p over β. We will show that S is weakly contractible by explicitly constructing both an initial object and a final object in S. Note that S, being a hom-space in a 2-category, is just a 1-category. So the notion of initial and final objects in S is the usual one.

Let $X_L = h_L \circ b$ and consider the following object in S which we denote C_L:

$$\begin{tikzcd}
 & C \arrow[rr, "f"] \arrow[dd, equals] & & D \arrow[dd, equals] \\
M \arrow[ur, "a"] \arrow[rr, "b"] \arrow[ddd, "X_L"'] & & B \arrow[ur, "g"] \arrow[dd, equals] & \\
 & C \arrow[rr, "f"] & & D \\
A \arrow[ur, "i"] \arrow[rr, "h"'] & & B \arrow[ur, "g"'] &
\end{tikzcd}$$

where the front face is given by the unit η of the adjunction $(h_L \dashv h)$ and the left face is given by the composition (reading from left to right)

$$\begin{array}{ccccccc}
\downarrow b & & \downarrow b & \overset{\beta}{\times} & \downarrow a & & \downarrow a \\
\downarrow h_L & \overset{\alpha_L^{-1}}{\times} & \downarrow g & & \downarrow f & \overset{\epsilon}{\times} & \| \\
\downarrow i & & \downarrow f_L & & \downarrow f_L & & \|
\end{array}$$

It can be easily checked that this cube commutes.

Similarly we can construct such a cube using $X_R = i_R \circ a$. These cubes are not in general isomorphic, but we have the following:

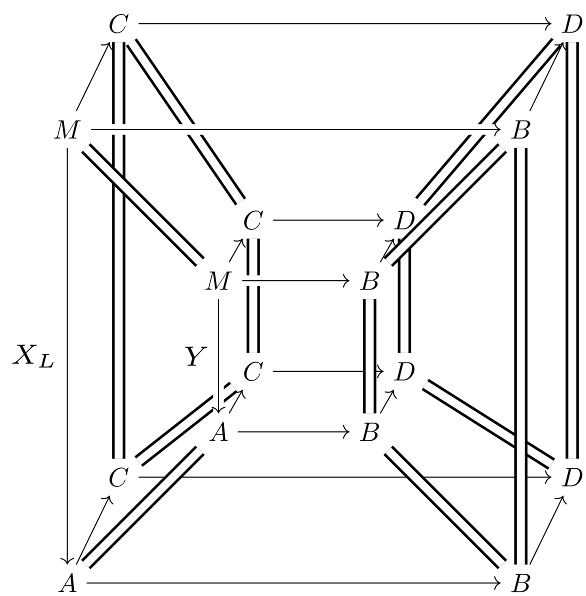
Claim. *The cube constructed with X_L is initial in S, and the cube constructed with X_R is final in S.*

Let us check it for X_L (for X_R it is almost identical).

Suppose that we have another element $C_Y \in S$, i.e. a cube

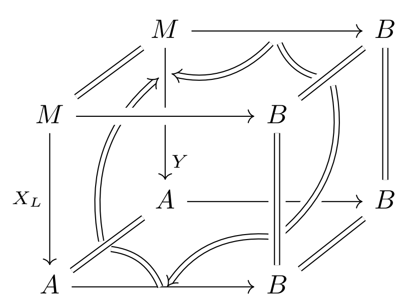
$$\begin{tikzcd}
 & C \arrow[rr, "f"] \arrow[dd, equals] & & D \arrow[dd, equals] \\
M \arrow[ur, "a"] \arrow[rr, "b"] \arrow[ddd, "Y"'] & & B \arrow[ur, "g"] \arrow[dd, equals] & \\
 & C \arrow[rr, "f"] & & D \\
A \arrow[ur, "i"] \arrow[rr, "h"'] & & B \arrow[ur, "g"'] &
\end{tikzcd}$$

A morphism $C_L \to C_Y$ in the fiber S is, by definition, a commutative 4-cube



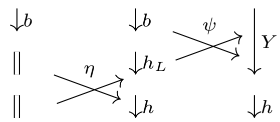
where the dimensions are ordered as [$\rightarrow$, $\downarrow$,radial towards center,$\nearrow$]. This unambiguously defines the orientation of 2-morphisms and the order of their composition - see Appendix A for details. The diagram of 2-morphisms associated to the oriented 4-cube commutes iff each sub 3-cube commutes (proof in [Gra76]). Hence, in all, the data we should provide is only one 2-face, namely the one shared by X_L and Y, and this just amounts to a morphism $X_L \xrightarrow{\psi} Y$. We need to show that such a morphism which makes the 4-cube commute exists and is unique. Note that this presentation of morphisms of S as a 4-cube coincides with the usual way in which the universality of the comma object is defined.

Consider the front 3-cube



By assumption it commutes. Its front face is the front face of C_L; its back face is the front face of C_Y, denote it by f_Y; its left face is ψ; the rest are degenerate.

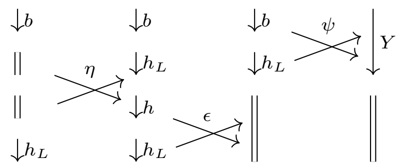
So we get the equation



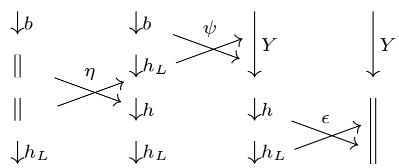
$$= f_Y$$

So since η is part of an adjunction, we have that

$$\psi =$$



$$=$$



$$= \epsilon \circ f_Y$$

where the second step is justified by the four-interchange law in a 2-category.

So ψ is uniquely defined, and all we need to check is that it makes the left cube in the 4-cube commute, which is a similar computation.

This completes the proof. $\square$

2.7.1 General comma squares in $\mathbf{FinSet}^{\sqcup}$ and $\mathbf{Cat}_{ex}^{\otimes}$

Definition 2.20. A map $f : [n]_* \to [m]_*$ in $\mathbf{FinSet}_*$ is called *active* if $f^{-1}\{*\} = \{*\}$.

Consider a square in $\mathbf{FinSet}^{\sqcup}$:

$$\begin{tikzcd} (A_i)_{i\in I} \arrow[r, "\varphi"] \arrow[d, "\psi"] & (B_j)_{j\in J} \arrow[d, "\tau"] \\ (C_k)_{k\in K} \arrow[r, "\nu"] & (D_l)_{l\in L} \end{tikzcd}$$

and suppose that it lies over a square of active maps in $\mathbf{FinSet}_*$. We say in this case that it is an *active* square.

In this section we want to prove:

Proposition 2.21. *An SSH functor $\mathcal{H}$ takes active comma squares in* $\mathbf{FinSet}^{\sqcup}$ *to comma squares in* $\mathbf{Cat}_{ex}^{\otimes}$.

Remark 2.22. The subcategory of active maps in $\mathbf{FinSet}_*$ is closed on pullbacks in $\mathbf{FinSet}_*$, so a comma square of active maps is just a usual comma square in the subcategory $(\mathbf{FinSet}_*)_{ac} \cong \mathbf{FinSet}$.

Lemma 2.23. *An active square is comma in* $\mathbf{FinSet}^{\sqcup}$*, iff the square it lies over,* $\begin{smallmatrix} [I]_* & \longrightarrow & [J]_* \\ \downarrow & & \downarrow \\ [K]_* & \longrightarrow & [L]_* \end{smallmatrix}$ *, is comma in* $\mathbf{FinSet}_*$*, and all the squares of sets involved in it are comma in* $\mathbf{FinSet}$*.*

Proof. We may assume that L has one point, and then we have an explicit construction for a comma square with given bottom right corner. Namely, we consider sets $T_{k,j}, k \in K, j \in J$ given by $T_{k,j} = B_j \times_D C_k$ with the obvious projections to $B_\bullet, C_\bullet$. The lemma then follows immediately. $\square$

An SSH functor, having as codomain the coCartesian symmetric monoidal category $\mathbf{FinSet}^{\sqcup}$, automatically factors through a functor $\bar{\mathcal{H}}$ to $\mathrm{CAlg}(\mathbf{Cat}_{ex})^{\otimes}$ (see [Lur11] §3.2.4.9, and also our Proposition 2.14). It is easy to check that the restriction $\bar{\mathcal{H}} : \mathbf{FinSet} \to \mathrm{CAlg}(\mathbf{Cat}_{ex})$ preserves comma squares (cf [Lur11] Corollary 3.2.2.5).

Remark 2.24. By Proposition 3.2.4.7 in [Lur11] the symmetric monoidal structure on $\mathrm{CAlg}(\mathbf{Cat}_{ex})$ is coCartesian. A consequence of this is that a map in $\mathrm{CAlg}(\mathbf{Cat}_{ex})^{\otimes}$ is a collection of maps as in the description of $\mathbf{FinSet}^{\sqcup}$ in 2.1.

Since the forgetful functor $\mathrm{CAlg}(\mathbf{Cat}_{ex})^{\otimes} \to \mathbf{Cat}_{ex}^{\otimes}$ has a left adjoint (see [Lur11] Example 3.1.3.12) it preserves all limits, in particular comma squares. So it is enough for us to prove

Proposition 2.25. *$\bar{\mathcal{H}}$ takes active comma squares in* $\mathbf{FinSet}^{\sqcup}$ *to comma squares in* $\mathrm{CAlg}(\mathbf{Cat}_{ex})^{\otimes}$.

Proof. Let

$$\begin{array}{ccc} (T_i)_{i\in I} & \longrightarrow & (S_j)_{j\in J} \\ \downarrow & & \downarrow \\ (U_k)_{k\in K} & \longrightarrow & V \end{array}$$

be an active comma square in $\mathbf{FinSet}^{\sqcup}$. As noted above, $I = J \times K$ and for each j, k, the square

$$\begin{array}{ccc} T_{j,k} & \longrightarrow & S_j \\ \downarrow & & \downarrow \\ U_k & \longrightarrow & V \end{array}$$

is a comma square. As noted earlier, its image

$$\begin{tikzcd}
\bar{\mathcal{H}}(T_{j,k}) \arrow[r] \arrow[d] & \bar{\mathcal{H}}(S_j) \arrow[d] \\
\bar{\mathcal{H}}(U_k) \arrow[r] & \bar{\mathcal{H}}(V)
\end{tikzcd} \tag{2.2}$$

is then also a comma square.

Consider a square

$$\begin{tikzcd}
(\mathcal{D}_i)_{i\in I} \arrow[r] \arrow[d] & \mathcal{H}((S_j)_{j\in J}) \arrow[d] \\
\mathcal{H}((U_k)_{k\in K}) \arrow[r] & \mathcal{H}(V)
\end{tikzcd}$$

in $\mathrm{CAlg}(\mathbf{Cat}_{ex})^{\otimes}$. We want to show that it has an essentially unique map to the square

$$\begin{tikzcd}
\mathcal{H}((T_i)_{i\in I}) \arrow[r] \arrow[d] & \mathcal{H}((S_j)_{j\in J}) \arrow[d] \\
\mathcal{H}((U_k)_{k\in K}) \arrow[r] & \mathcal{H}(V)
\end{tikzcd}$$

By restricting to each pair j, k (see 2.24) we get squares

$$\begin{tikzcd}
\mathcal{D}_{j,k} \arrow[r] \arrow[d] & \mathcal{H}(S_j) \arrow[d] \\
\mathcal{H}(U_k) \arrow[r] & \mathcal{H}(V)
\end{tikzcd}$$

and these have essentially unique maps to the comma squares (2.2), which then give a map of the original squares. $\square$

Remark 2.26. It seems to be possible to give a more elegant proof of the fact that H preserves all comma squares (and not just active ones) using the equivalence of categories between the cateogory of morphisms of ∞-operads $\mathbf{FinSet}^{\sqcup} \to \mathrm{CAlg}(\mathbf{Cat}_{ex})^{\otimes}$ and the category of functors $\mathbf{FinSet} \to \mathrm{CAlg}(\mathbf{Cat}_{ex})$ (Proposition 2.4.3.8 in [Lur11]) and the results of [KL00] adapted for the ∞-operads setting.

3 The category $\mathcal{P}$ of polynomial functors

3.1 Recollection of $\mathcal{P}$ and its Grothendieck group

We consider the category $\mathcal{P}$ of polynomial functors over a field $\Bbbk$ of characteristic 0, defined by Friedlander and Suslin in [FS97].

Definition 3.1. The category of polynomial functors $\mathcal{P}$ is the category whose objects are functors from **Vect** to **Vect** that induce polynomial maps on the Hom spaces, i.e.

$$F : \mathbf{Vect} \to \mathbf{Vect}$$

such that for any two spaces V, W, the map

$$\mathrm{Hom}_{\Bbbk}(V, W) \to \mathrm{Hom}_{\Bbbk}(FV, FW)$$

is a polynomial map.

Definition 3.2. For any finite set S we consider the category $\mathcal{P}^{\otimes S}$ of polynomial functors from the category $\mathrm{Sh}(S)$ of sheaves of vector spaces over S to **Vect**.

It is easy to check that $\mathcal{P}^{\otimes S}$ satisfies the universal property of the tensor product of categories in $\mathbf{Cat}_{ex}$ as defined by Deligne in [Del90] (see Definition 2.9 for details). Namely, for any set S we have the functor

$$\boxtimes_S : \mathcal{P}^{\times S} \to \mathcal{P}^{\otimes S} \qquad \forall V \in \mathrm{Sh}(S)(\boxtimes_S F_s)(V_s) = \otimes_S F_s(V_s) \tag{3.1}$$

that presents $\mathcal{P}^{\otimes S}$ as the Deligne tensor of $\mathcal{P}^{\times S}$.

Proposition 3.3. *The Grothendieck K-group of $\mathcal{P}$ is isomorphic to Λ.*

Proof. As is shown in [FS97], the subcategory of polynomial functors of degree $\leq d$ is equivalent to the category of polynomial representations of GL_n of degree $\leq d$, when $n \geq d$. Sending a representation to its character on the torus gives an isomorphism of the K group of this subcategory with the $\mathbb{Z}$-group of symmetric polynomials in n variables of degree $\leq d$. Going to the limit finishes the proof. $\square$

Corollary 3.4. *The Grothendieck K-group of $\mathcal{P}^{\otimes S}$ is isomorphic to $\Lambda^{\otimes S}$.*

3.2 The SSH structure on $\mathcal{P}$

In this section we put a SSH structure on $\mathcal{P}$, in the sense of §2, i.e. we construct a symmetric monoidal functor $\mathbf{FinSet}^{\sqcup} \to \mathbf{Cat}_{ex}^{\otimes}$.

Proposition 3.5. *The following collection of data gives a SSH structure on $\mathcal{P}$.*

- *To any finite set S, we assign the category $\mathcal{P}^{\otimes S}$ defined above.*
- *To any map of finite sets $\varphi : S \to T$ we assign the functor*

  $$m_\varphi : \mathcal{P}^{\otimes S} \to \mathcal{P}^{\otimes T}$$

  defined by the formula

  $$m_\varphi(F) := F \circ \varphi^*$$

  note that it has a natural adjoint (both left and right),Δ_φ, given by

  $$\Delta_\varphi(\Phi) := \Phi \circ \varphi_*$$

  and hence is in $\mathbf{Cat}_{ex}$

- *To any map $(S_i) \xrightarrow{(\varphi_i)} T$ in $\mathbf{FinSet}^{\sqcup}$, over the active map $[n]_* \to [1]_*$, we assign the functor*

$$m_{(\varphi_i)} : \prod \mathcal{P}^{\otimes S_i} \to \mathcal{P}^{\otimes T}$$

given by

$$m_{(\varphi_i)}(F_i)(V) = \bigotimes F_i(\varphi_i^* V)$$

- *To any commutative square of sets we associate the 2-commutative diagram*

$$\begin{tikzcd} S \arrow[r, "a"] \arrow[d, "c"] & T \arrow[d, "b"] \\ R \arrow[r, "d"] & U \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{P}^{\otimes S} \arrow[r, "m_a"] \arrow[d, "m_c"] \arrow[dr, phantom, "\swarrow\!\!\!\!\!\sim"] & \mathcal{P}^{\otimes T} \arrow[d, "m_b"] \\ \mathcal{P}^{\otimes R} \arrow[r, "m_d"] & \mathcal{P}^{\otimes U} \end{tikzcd}$$

with an isomorphism in the middle which comes from the isomorphims $\varphi^*\psi^* \cong (\psi\varphi)^*$.

Proof. Denote by $\mathcal{H}$ the resulting functor.

First, we have explicit adjunctions $(m_\varphi \dashv \Delta_\varphi)$ and $(\Delta_\varphi \dashv m_\varphi)$ given by the explicit adjunctions $(\varphi_* \dashv \varphi^*)$ and $(\varphi^* \dashv \varphi_*)$(see Appendix C). So $\mathcal{H}$ lands in $\mathbf{Cat}_{ex}$. That it is symmetric monoidal follows from (3.1).

Regarding the other conditions:

Obviously, $\mathcal{H}(\emptyset)$ is canonically equivalent to **Vect**, and $\mathcal{H}([1])$ is canonically equivalent to $\mathcal{P}$.

It remains to show that $\mathcal{H}$ sends Cartesian squares in the fiber over $[1]_*$ to squares satisfying the BC condition. Consider a Cartesian square of maps of sets and the corresponding diagram of tensor powers

$$\begin{tikzcd} S \arrow[r, "a"] \arrow[d, "c"] & T \arrow[d, "b"] \\ R \arrow[r, "d"] & U \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{P}^{\otimes S} \arrow[r, "m_a"] \arrow[d, "m_c"] \arrow[dr, phantom, "\swarrow\!\!\!\!\!\sim"] & \mathcal{P}^{\otimes T} \arrow[d, "m_b"] \\ \mathcal{P}^{\otimes R} \arrow[r, "m_d"] & \mathcal{P}^{\otimes U} \end{tikzcd}$$

Since the funtors m_φ were defined as precomposition with φ^*, it is enough to show that the square

$$\begin{tikzcd} \mathrm{Sh}(S) & \mathrm{Sh}(T) \arrow[l, "a^*"'] \\ \mathrm{Sh}(R) \arrow[u, "c^*"] \arrow[ur, phantom, "\swarrow\!\!\!\!\!\sim"] & \mathrm{Sh}(U) \arrow[l, "d^*"] \arrow[u, "b^*"'] \end{tikzcd}$$

satisfies the Beck-Chevalley condition, but this follows immediately from proper base change, so we are done.

□

4 The Fock space action for positive self-adjoint Hopf algebras

4.1 The Fock space action

Given a pair of Hopf algebras and a pairing between them one can form an algebra called the *Heisenberg double* and construct a representation of it called the Fock space representation. The description of the general notion can be found, for example, in [Kap98]. In this article we are concerned with the specific case of positive selfadjoint Hopf algebras introduced by Zelevinsky in [Zel81]. Let us recall the definitions:

Definition 4.1. A positive selfadjoint Hopf (PSH) algebra is a graded connected Hopf algebra over $\mathbb{Z}$ with an inner product and a distinguished finite orthogonal $\mathbb{Z}$ basis in each grade s.t. multiplication and comultiplication are adjoint and take elements with positive coefficients to elements with positive coefficients.

Definition 4.2. A graded Hopf algebra over $\mathbb{Z}$ $A = \bigoplus_{n\geq 0} A_n$ is called *connected* if the unit morphism and the counit morphism restricted to A_0 give an isomorphism of A_0 with $\mathbb{Z}$

For a positive selfadjoint Hopf algebra A we consider the dual pair (A, A) and outline the construction of the Heisenberg double and the Fock space representation in this case. We will use the construction in this section to show that the SSH structure on a category gives rise to its categorical analog in Section 5. Consider a PSH algebra A and denote the adjoint multiplication and comultiplication maps by

$$m : A \otimes A \to A \qquad\qquad \Delta : A \to A \otimes A$$

Note that A, like any PSH algebra, is commutative and cocommutative, a result proven in [Zel81].

For each $x \in A$ we define operators $m_x, \Delta_x : A \to A$ by the formulas

$$m_x = m \circ i_x \quad \Delta_x = j_x \circ \Delta \tag{4.1}$$

where

$$i_x(y) = x \otimes y \qquad\qquad j_x(y \otimes z) = z < x, y >$$

Remark 4.3. Note that m_x, Δ_x are adjoint for any $x \in A$.

We use these operators to define a morphism of $\mathbb{Z}$ groups

$$\varphi : A \otimes A \to \mathrm{End}_{\mathbb{Z}}(A) \qquad\qquad x \otimes y \mapsto m_x \Delta_y$$

Proposition 4.4. *φ is injective and its image is a subalgebra of $End_{\mathbb{Z}}(A)$*

Notation 4.5. Since φ is injective, it induces an algebra structure on $A \otimes A$. We denote the algebra $A \otimes A$, with the algebra structure given by φ, by $\mathrm{Heis}(A)$. The natural action of $\mathrm{Heis}(A)$ on A is called the *Fock space action*

Proof. To prove that the image of φ is a subalgebra note the we have the following relations $\forall x, y \in A$:

$$m_x m_y = m_{m(xy)} = m_{m(yx)} = m_y m_x \tag{4.2}$$
$$\Delta_x \Delta_y = \Delta_{yx} = \Delta_{xy} = \Delta_y \Delta_x \tag{4.3}$$
$$\Delta_x m_y = m\Delta^2_{\Delta(x)} i_y \tag{4.4}$$

where if $\Delta(x) = x_{(1)} \otimes x_{(2)}$ (in Sweedler notation) then

$$\Delta^2_{\Delta(x)} i_y(z) = \Delta^2_{\Delta(x)}(y \otimes z) := \Delta_{x_{(1)}} y \otimes \Delta_{x_{(2)}} z \tag{4.5}$$

The first two relations hold since multiplication and comultiplication in A are associative and commutative and the relation 4.4 holds since (as shown in [Zel81]) $\forall z, u \in A$:

$$\begin{aligned} &< \Delta_x m_y z, u >=< \Delta_x m(y \otimes z), u >= \\ &\quad < m(y \otimes z), m(x \otimes u) >=< y \otimes z, \Delta m(x \otimes u) >= \\ &\quad < y \otimes z, m(\Delta x \otimes \Delta u) >=< y \otimes z, m((x_{(1)} \otimes x_{(2)}) \otimes \Delta u) >= \\ &\quad < \Delta_{x_{(1)}} y \otimes \Delta_{x_{(2)}} z, \Delta u >=< m(\Delta_{x_{(1)}} y \otimes \Delta_{x_{(2)}} z), u > \end{aligned}$$

(we used the fact that $m_{x_{(1)} \otimes x_{(2)}}$ is adjoint to $\Delta_{x_{(1)} \otimes x_{(2)}}$ on $A^{\otimes 2}$)
So explicitly, the third relation gives us that

$$\Delta_x m_y = m_{\Delta_{x_{(2)}}(y)} \Delta_{x_{(1)}} \in \varphi(A \otimes A)$$

To prove that φ is injective we use the fact that A is graded and each grade has an orthogonal basis. Let $\sum_i x_i y_i \in Ker\varphi$ and assume without loss of generality that y_i are the elements of the orthogonal basis of A. Let y_{i_0} be an element of minimal degree so that $x_{i_0} \neq 0$ and let $r = \deg y_{i_0}$. It follows from the definition of Δ_y that for every $z \in A_n, y \in A_m$ $\Delta_y z = 0$ when $n < m$, and $\Delta_y z =< y, z >$ for $n = m$. Hence by applying the operator $\sum_i m_{x_i} \Delta_{y_i}$ to y_{i_0} we get $x_{i_0} = 0$, a contradiction. □

To construct the categorification of the Fock space action in §5 we will use the following

Observation. The relation $\Delta_x m_y = m\Delta^2_{\Delta(x)} i_y$ for any x, y is equivalent to the relation $\Delta_x m = m\Delta^2_{\Delta(x)}$ for any x.

The above construction can be easily generalized to give the following general statement:

Proposition 4.6. *Giving an action of* $\mathrm{Heis}(A)$ *on a space* V *is the same as giving a morphism of spaces* $A \otimes A \xrightarrow{a} \mathrm{End}(V)$ *which satisfies*

1. *The restrictions of a to $A \otimes 1$ and $1 \otimes A$ are morphisms of algebras.*
2. *Denote*

$$\Delta_a^2 : A \otimes A \to \mathrm{End}(A \otimes V)$$
$$\Delta_a^2(x \otimes y)(z \otimes v) := \Delta_x(z) \otimes (a(1 \otimes y)(v))$$
$$m_a : A \otimes V \to V$$
$$m_a(z \otimes v) := a(z \otimes 1)(v)$$

then for any $x \in A$ we have $a(1 \otimes x) \circ m_a = m_a \circ \Delta_a^2(\Delta(x))$

4.2 The infinite-dimensional Heisenberg algebra

The classical one-variable Heisenberg algebra is the $\mathbb{Z}$-algebra with two generators p, q and one defining relation $[p, q] = 1$. The infinite version of this algebra is usually defined in terms of infinite number of generators and relations. There are several different versions used in different settings, some of which are described below. All of these are (sometimes non-isomorphic) $\mathbb{Z}$ forms of the same complex algebra.

We use the notion of of Heisenberg double for a positive self-adjoint Hopf algebra which we explored in the previous section to give an alternative description of the infinite-dimensional Heisenberg algebra. This description doesn't use the language of generators and relations, and thus lends itself more naturally to categorification.

Denote by Λ the algebra of symmetric polynomials in an infinite number of variables over $\mathbb{Z}$. Λ has the structure of a PSH algebra. The $\mathbb{Z}$ basis is given by the Schur polynomials; this also defines an inner product. The multiplication map m is given by the multiplication of polynomials.

Definition 4.7. We define the Heisenberg algebra of infinite rank to be $\mathrm{Heis}(\Lambda)$, i.e. the the Heisenberg double corresponding to the pair (Λ, Λ).

Let us describe how some of the commonly used definitions of Heisenberg algebra arise from the above definition:

1. A $\mathbb{Z}$-algebra with generators $p_n, q_n, n \in \mathbb{N}$ and relations
   - $[p_m, p_n] = [q_m, q_n] = 0$
   - $[p_m, q_n] = \delta_{mn} 1$

   This corresponds to taking p_n to be the elementary symmetric function of degree n ($\sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$) in the left Λ and the q_n to be the primitive symmetric function of degree n (described in [Zel81]) in the right Λ.

2. A $\mathbb{Z}$-algebra with generators $c_k, k \in \mathbb{Z}$ and relations $[c_k, c_l] = k\delta_{k+l,0}$. This corresponds to taking c_k with positive k to be the primitive symmetric function of degree k in the right Λ and c_k with negative k to be the primitive symmetric function of degree k in the left Λ, and taking $c_0 = 1$.

3. (The algebra which Khovanov categorifies in [Kho10]) Generators a_n, b_n, $n \in \mathbb{N}$ with relations

   - $a_0 = b_0 = 1$
   - $[a_m, a_n] = [b_m, b_n] = 0$
   - $[a_m, b_n] = b_{n-1}a_{m-1}$

   This corresponds to taking a_n to be the elementary symmetric function of degree n in the left Λ and the b_n to be the whole symmetric function of degree n ($\sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}$) in the right Λ.

5 The categorification of the Fock space action

In this section we describe a categorification of the Fock space representation of the Heisenberg double $\mathrm{Heis}(A)$. In section 4 we described how this representation is constructed using the PSH algebra structure on A. Presently from a self-adjoint Hopf category structure on a category $\mathcal{C}$ we will construct a categorical action on $\mathcal{C}$ which descends to the Fock space representatiof $\mathrm{Heis}(K(\mathcal{C}))$.

In particular, we show how the self-adjoint Hopf category structure on the category $\mathcal{P}$ of polynomial functors gives rise to categorification of the Fock space action of the Heisenberg algebra $\mathrm{Heis}(\Lambda)$.

5.1 The statement

Denote the left and right adjoints of the functor $m : \mathcal{C}^{\otimes[2]} \to \mathcal{C}$ by Δ^l and Δ^r. For the straightforward categorification of Proposition 4.6 in the case of the Fock space action we need the following:

1. Endofunctors m_F, Δ^r_F of $\mathcal{C}$, for any $F \in \mathcal{C}$, and endofunctors $(\Delta^r)^2_\Phi$ of $\mathcal{C}^{\otimes[2]}$ for any $\Phi \in \mathcal{C}^{\otimes[2]}$.
2. Isomorphisms $m_F m_G \cong m_{F\otimes G}$ and $\Delta^r_G \Delta^r_F \cong \Delta^r_{F\otimes G}$.
3. Isomorphisms $\Delta^r_F m \cong m(\Delta^r)^2{}_{\Delta^l(F)}$

The roles of Δ^l and Δ^r here and in what follows may be reversed; they both descend to the same map in the K-group.

In the view of the Proposition 4.6 this would give us

Theorem 2. *Denote by* $\mathrm{End}_{ex}(\mathcal{C})$ *the category of exact endofunctors of* $\mathcal{C}$*. The functor* $\mathcal{C} \otimes \mathcal{C}^{op} \to \mathrm{End}_{ex}(\mathcal{C})$ *given by* $F \boxtimes G \mapsto m_F \circ \Delta^r_G$ *descends to the Fock space representation of* $\mathrm{Heis}(K(\mathcal{C}))$*.*

In the following section we will use the self-adjoint Hopf structure on $\mathcal{C}$ to construct the adjoint functors i_F, j^r_F, j^l_F and define $m_F = m \circ i_F, \Delta^r_F = j^r_F \circ \Delta^r, \Delta^l_F = j^l_F \circ \Delta^l$, and similarly for the extensions to $\mathcal{C}^{\otimes[2]}$. We will then show that the 2-morphisms in question are constructed naturally from the SSH structure. Their invertibility is a continuity statement (see Proposition 2.11), which follows from the continuity of the SSH functor.

5.2 Constructions

Consider the functor $\boxtimes : \mathcal{C}^{\times 2} \to \mathcal{C}^{\otimes[2]}$ that is the image under $\mathcal{H}$ of the coCartesian arrow $(\{1\},\{2\}) \xrightarrow{+} \{1,2\}$. We define for any object $F \in \mathcal{C}$, the functor i_F as

$$i_F : \mathcal{C} \to \mathcal{C}^{\otimes[2]}, \qquad\qquad i_F(X) = F \boxtimes X$$

Note that by our requirements i_F has left and right adjoints which we denote by j^l_F, j^r_F

$$j^l_F, j^r_F : \mathcal{C}^{\otimes[2]} \to \mathcal{C}$$

We define $m_F, \Delta^l_F, \Delta^r_F$ by

$$m_F = m \circ i_F \qquad\qquad \Delta^r_F = j^r_F \circ \Delta^r \qquad\qquad \Delta^l_F = j^l_F \circ \Delta^l$$

Δ^r_F, Δ^l_F are respectively right and left adjoint to m_F for any $F \in \mathcal{C}$.

To construct the isomorphism $m_F m_G \cong m_{F\otimes G}$ we first note that the functor $m_{F\otimes G}$ is canonically isomorphic to the functor $m_{m_F G}$ - the isomorphism $F \otimes G \cong m_F G$ is a part of the self-adjoint Hopf structure on $\mathcal{C}$ as explained in §2. The functors $m_F m_G$ and $m_{m_F G}$ are the images of the following morphisms in $\mathbf{FinSet}^{\sqcup}$ under $\mathcal{H}$:

$$m_F m_G H : \qquad \begin{tikzcd}
(F)[1] \arrow[r] & {[1]} \arrow[r] & {[1]} \arrow[dr] & & \\
(G)[1] \arrow[dr] & & & {[2]} \arrow[r] & {[1]} \\
& {[2]} \arrow[r] & {[1]} \arrow[ur] & & \\
(H)[1] \arrow[ur] & & & &
\end{tikzcd}$$

$$m_{m_F G} H : \qquad \begin{tikzcd}
(F)[1] \arrow[dr] & & & & \\
& {[2]} \arrow[r] & {[1]} \arrow[dr] & & \\
(G)[1] \arrow[ur] & & & {[2]} \arrow[r] & {[1]} \\
(H)[1] \arrow[r] & {[1]} \arrow[r] & {[1]} \arrow[ur] & &
\end{tikzcd}$$

These morphisms form a commutative diagram in $\mathbf{FinSet}^{\sqcup}$, hence we have an isomorphism $m_F m_G \cong m_{m_F G}$ whenever we have a self-adjoint Hopf structure on $\mathcal{C}$.

Passing to adjoints we get an isomorphism

$$\Delta^r_{F\otimes G} \cong \Delta^r_G \Delta^r_F$$

It remains to give a categorical analog of the relation $\Delta_x m = m(\Delta)^2{}_{\Delta(x)}$. Let $\bar{c}$ be the arrow $([2],[2]) \to ([4])$ in $\mathbf{FinSet}^{\sqcup}$ given by the maps $\substack{1\mapsto 1\\ 2\mapsto 3}, \substack{1\mapsto 2\\ 2\mapsto 4}$. For $\Phi \in \mathcal{C}^{\otimes[2]}$ we define the functor $i_\Phi : \mathcal{C}^{\otimes[2]} \to \mathcal{C}^{\otimes[4]}$ by

$$i_\Phi(X) = \mathcal{H}(\bar{c})(\Phi, X)$$

Define adjoint functors $m^2_\Phi \dashv \Delta^2_\Phi$ as follows:

$$\begin{aligned} m^2_\Phi &= m^2 \circ i_\Phi \\ (\Delta^r)^2_\Phi &= j^r_\Phi \circ (\Delta^r)^2 \end{aligned}$$

then in this notation we have

Theorem 3. *There is a canonical isomorphism* $\Delta^r_F m \cong m \circ (\Delta^r)^2_{\Delta^l(F)}$ *coming from the SSH structure on* $\mathcal{C}$*.*

Corollary 5.1. *We have a canonical isomorphism*

$$\Delta^r_F m_G = \Delta^r_F m \circ i_G \cong m \circ (\Delta^r)^2_{\Delta(F)} \circ i_G$$

To construct the isomorphism from Theorem 3 note first that this isomorphism can be represented as a square

$$\begin{tikzcd} \mathcal{C}^{\otimes 2} \arrow[r, "m"] & \mathcal{C} \\ \mathcal{C}^{\otimes 2} \arrow[u, "(\Delta^r)^2_{\Delta^l(F)}"] \arrow[r, "m"'] \arrow[ur, phantom, "\cong"] & \mathcal{C} \arrow[u, "\Delta^r_F"'] \end{tikzcd} \tag{5.1}$$

The form of this square suggests that we should try to construct it as the right mate of a square of the form

$$\begin{tikzcd} \mathcal{C}^{\otimes 2} \arrow[r, "m"] \arrow[d, "m^2_{\Delta^l(F)}"'] \arrow[dr, phantom, "\cong"] & \mathcal{C} \arrow[d, "m_F"] \\ \mathcal{C}^{\otimes 2} \arrow[r, "m"'] & \mathcal{C} \end{tikzcd} \tag{5.2}$$

and then check that it is invertible. Such a square is explicitly constructed from the SSH structure by considering a composition of squares:

$$\begin{tikzcd} \mathcal{C}^{\otimes 2} \arrow[r, "m"] \arrow[d, "i_{\Delta^l(F)}"'] \arrow[dr, phantom, "\Swarrow"] & \mathcal{C} \arrow[d, "i_F"] \\ \mathcal{C}^{\otimes 4} \arrow[r, "\overline{m}"] \arrow[d, "m^2"'] \arrow[dr, phantom, "\Swarrow"] & \mathcal{C}^{\otimes 2} \arrow[d, "m"] \\ \mathcal{C}^{\otimes 2} \arrow[r, "m"] & \mathcal{C} \end{tikzcd}$$

where $\overline{m}$ corresponds to the maps of sets $\substack{1\mapsto 1 \quad 2\mapsto 2 \\ 3\mapsto 1,\, 4\mapsto 2}$.

The lower square is the image of a comma square under the SSH functor, and therefore its right mate is invertible. In §5.3 we construct the upper square α' from the unit of the adjunction: $F \to m\Delta^l F$ and show that it satisfies BC.

In §5.4 we spell out what this construction entails in the case of polynomial functors.

5.3 Categorical Heisenberg double

Note. Throughout this section we will write Δ for Δ^l - the left adjoint of the multiplication.

Recall that the SSH structure is given by a map of fibrations over $\mathbf{FinSet}_*$

$$\mathcal{H} : \mathbf{FinSet}^{\sqcup} \to \mathbf{Cat}_{ex}^{\otimes}$$

We can restrict $\mathcal{H}$ over the subcategory of active maps in $\mathbf{FinSet}_*$ to get a functor

$$\mathcal{H}_{ac} : (\mathbf{FinSet}^{\sqcup})_{ac} \to (\mathbf{Cat}_{ex}^{\otimes})_{ac}$$

Let $\mathbf{Cat}_{ex} \xrightarrow{i} (\mathbf{Cat}_{ex}^{\otimes})_{ac}$ be the imbedding of the (pseudo) fiber over $[1]_*$, that is a comma object defined by the square

$$\begin{tikzcd}
\mathbf{Cat}_{ex} \arrow[r] \arrow[d, "i"] & \{[1]_*\} \arrow[d] \\
(\mathbf{Cat}_{ex}^{\otimes})_{ac} \arrow[r] & (\mathbf{FinSet}_*)_{ac}
\end{tikzcd}$$

in the $(\infty,1)$-category of categories (i.e. we only consider the invertible 2-morphisms)

Definition 5.2. We define the *categorical Heisenberg double* - $\mathrm{Heis}(\mathcal{H})$, of an SSH functor $\mathcal{H}$, to be the comma object defined by the square

$$\begin{tikzcd}
\mathrm{Heis}(\mathcal{H}) \arrow[r] \arrow[d, "p"'] & \mathbf{Cat}_{ex} \arrow[d, "i"] \arrow[dl, Rightarrow, "\alpha"'] \\
(\mathbf{FinSet}^{\sqcup})_{ac} \arrow[r, "\mathcal{H}_{ac}"] & (\mathbf{Cat}_{ex}^{\otimes})_{ac}
\end{tikzcd} \tag{5.3}$$

(note that this square is a comma square in the $(\infty,2)$-category of categories.)

Proposition 5.3. *The functors out of* $\mathrm{Heis}(\mathcal{H})$ *preserve comma squares.*

Proof. Note first that $\mathcal{H}_{ac}$ preserves comma squares by Proposition 2.21. Replace $\mathbf{Cat}_{ex}$ with the category $\mathbf{Cat}_{\infty}^{\mathrm{Ex}}$ of stable infinity categories by composing with the canonical embedding (see Remark 2.3). Appealing to [BKP89] aplied to [KL00], we get that $\mathrm{Heis}(\mathcal{H})$ has all comma objects and the functors out of it preserve comma objects. This uses the fact that $\mathbf{Cat}_\infty^{\mathrm{Ex}}$, as opposed to $\mathbf{Cat}_{ex}$, has all comma objects.

Now we can restrict back to $\mathbf{Cat}_{ex}$ to get the square (5.3), and the property of preserving comma objects is obviously retained. $\square$

Remark 5.4. After restricting to $\mathbf{Cat}_{ex}$, $\mathrm{Heis}(\mathcal{H})$ no longer necessarily has all comma objects. This is another indication that $\mathbf{Cat}_\infty^{\mathrm{Ex}}$ is a better object to work with if one wants to get further results.

Using the standard construction for the comma square in $\mathbf{Cat}$, $\mathrm{Heis}(\mathcal{H})$ is the category described as follows:

- Objects are pairs $(C, S = (S_k)_{k\in K})$ with $C \in \mathbf{Cat}_{ex}$ and $S \in \mathbf{FinSet}^{\sqcup}$, along with a morphism $i(C) \xrightarrow{a} \mathcal{H}(S)$.
- A morphism $(C,S) \to (D,T)$ is a pair $(C \xrightarrow{\varphi} D, S \xrightarrow{f} T)$, and a square in $\mathbf{Cat}_{ex}^{\otimes}$

$$\begin{tikzcd} i(C) \arrow[r, "i(\varphi)"] \arrow[d] \arrow[dr, phantom, "\swArrow"] & i(D) \arrow[d] \\ \mathcal{H}(S) \arrow[r, "\mathcal{H}(f)"'] & \mathcal{H}(T) \end{tikzcd}$$

- And so on...

More explicitly, Let $(\mathcal{C}, (S_i), a)$ be an object in $\mathrm{Heis}(\mathcal{H})$. The map a sits over an injection $[1]_* \to [n]_*$ which sends 1 to k. Let $\mathcal{H}(S) = (\mathcal{H}(S_1)\dots\mathcal{H}(S_n))$, then a amounts to a choice of elements in all but the k^{th} category, and a map $\mathcal{C} \to \mathcal{H}(S_k)$.

Our goal is to find a comma square in $\mathrm{Heis}(\mathcal{H})$ which will go to a square

$$\begin{tikzcd} \mathcal{H}([2]) \arrow[r, "m"] \arrow[d, "i_{\Delta(F)}"'] \arrow[dr, phantom, "\swArrow"] & \mathcal{H}([1]) \arrow[d, "i_F"] \\ \mathcal{H}([4]) \arrow[r, "\overline{m}"'] & \mathcal{H}([2]) \end{tikzcd} \tag{5.4}$$

in $\mathbf{Cat}_{ex}$ since then it will automatically be a comma square, and hence will satisfy the BC condition.

Another way to see it, is that we should try to compute the product of the two maps in the bottom right corner as maps in $\mathrm{Heis}(\mathcal{H})$.

Namely, we have the following obvious preimages for the maps $i_F, \overline{m}$ (abbreviating $([k])$ for $\mathcal{H}([k])$):

$$\begin{tikzcd} ([1]) \arrow[r, "i_F"] \arrow[d, "(F,\mathrm{Id})"'] \arrow[dr, phantom, "\swArrow"] & ([2]) \arrow[d, equal] \\ ([1],[1]) \arrow[r, "\mathcal{H}(c)"'] & ([2]) \end{tikzcd} \qquad \begin{tikzcd} ([4]) \arrow[r, "\overline{m}"] \arrow[d, equal] \arrow[dr, phantom, "="] & ([2]) \arrow[d, equal] \\ ([4]) \arrow[r, "\overline{m}"] & ([2]) \end{tikzcd}$$

where $c = (1 \mapsto 1, 1 \mapsto 2)$ and in the left square the 2-morphism is the canonical morphism of the form $\mathrm{Id} \circ (f \circ g) \to f \circ g$.

So we consider the following bottom right corner in $\mathrm{Heis}(\mathcal{H})$:

$$
\begin{tikzcd}
 & & & ([1],[1]) \arrow[dd] \\
 & & ([1]) \arrow[ur, "{(F,\mathrm{Id})}"] \arrow[dd, "i_F"' near start] & \\
 & ([4]) \arrow[rr, "\overline{m}"] \arrow[dl, equal] & & ([2]) \\
([4]) \arrow[rr, "\overline{m}"] & & ([2]) \arrow[ur, equal] &
\end{tikzcd}
$$

And we want to complete it to a comma square. From Proposition 5.3 it follows that the back face should come from a comma square in $\mathbf{FinSet}^{\sqcup}$, so first we compute the product in the back (as in Lemma 2.23) to get

$$
\begin{tikzcd}
 & ([2],[2]) \arrow[rr, "{(m,m)}"] \arrow[dd, "(\bar{c})"'] & & ([1],[1]) \arrow[dd, "(c)"] \\
 & & ([1]) \arrow[ur, "{(F,\mathrm{Id})}"] \arrow[dd, "i_F"' near start] & \\
 & ([4]) \arrow[rr, "\overline{m}"] \arrow[dl, equal] & & ([2]) \\
([4]) \arrow[rr, "\overline{m}"] & & ([2]) \arrow[ur, equal] &
\end{tikzcd}
$$

where $\bar{c} = \left(\begin{smallmatrix} 1\mapsto 1 & 1\mapsto 2 \\ 1\mapsto 3, & 1\mapsto 4 \end{smallmatrix}\right)$. The back face is the image of an active comma square under $\mathcal{H}$, so is a comma square in $\mathbf{Cat}_{ex}$ by Proposition 2.21.

Next we compute the product on top. Consider the corner

$$
\begin{tikzcd}
 & ([1]) \arrow[d, "{(F,\mathrm{Id})}"] \\
([2],[2]) \arrow[r, "{(m,m)}"] & ([1],[1])
\end{tikzcd}
$$

Since the element in the bottom right is over $[2]_*$, a square with this corner is given by two squares, with corners:

$$\begin{tikzcd} & ([1]) \arrow[d, equal] \\ ([2]) \arrow[r, "(m)"] & ([1]) \end{tikzcd} \qquad\qquad \begin{tikzcd} & () \arrow[d, "(F)"] \\ ([2]) \arrow[r, "(m)"] & ([1]) \end{tikzcd}$$

and we should compute the product separately for each one. The first corner fits in a degenerate square, and any degenerate square is comma. For the second corner we prove:

Lemma 5.5. *The square*

$$\begin{tikzcd} () \arrow[r, equal] \arrow[d, "(\Delta(F))"'] & () \arrow[d, "(F)"] \arrow[dl, phantom, "\Swarrow\eta"] \\ ([2]) \arrow[r, "m"] & ([1]) \end{tikzcd}$$

with $\eta : F \to m\Delta(F)$ coming from the adjunction, is comma.

Assuming the Lemma, we have

$$\begin{tikzcd} & ([2],[2]) \arrow[rr, "(m{,}m)"] \arrow[dd, "(\bar{c})"'] & & ([1],[1]) \arrow[dd, "(c)"] \\ ([2]) \arrow[ur, "(\Delta(F){,}\mathrm{Id})"] \arrow[rr, "m"] & & ([1]) \arrow[ur, "(F{,}\mathrm{Id})"] \arrow[dd, "i_F"] & \\ & ([4]) \arrow[rr, "\overline{m}"] & & ([2]) \\ ([4]) \arrow[ur, equal] \arrow[rr, "\overline{m}"] & & ([2]) \arrow[ur, equal] & \end{tikzcd}$$

and we complete to a cube by setting the missing 1-morphism to be the composition $(\bar{c}) \circ (\Delta(F), \mathrm{Id}) = i_{\Delta(F)}$; the left face to be the canonical isomorphism to the composition; and the front face to be the unique 2-morphism that makes the cube commute.

Lemma 5.6. *Consider a cube in a 2-category:*

$$\begin{tikzcd} & {} \arrow[rr, "x"] \arrow[dd, "y"'] & & {} \arrow[dd, "y"] \\ {} \arrow[ur, "z"] \arrow[rr, "x"] \arrow[dd, "y"'] & & {} \arrow[ur, "z"] \arrow[dd, "y"] & \\ & {} \arrow[rr, "x"] & & {} \\ {} \arrow[ur, "z"] \arrow[rr, "x"] & & {} \arrow[ur, equal] & \end{tikzcd}$$

with all faces given except the front, and such that the left and bottom faces are invertible, then there is a unique 2-morphism that fits in the front face and makes the cube commute.

Proof. The graph of 2-morphisms is

$$\begin{tikzcd}
& yx \arrow[r, "\sim"] & yzx \arrow[dr, "\sim"] & \\
xy \arrow[ur, "\alpha"] \arrow[dr] & & & zyx \\
& xzy \arrow[r] & zxy \arrow[ur] &
\end{tikzcd}$$

so the morphism $\alpha : xy \to yx$ that makes this diagram commute exists and is unique. ☐

In all we have the cube (representing a comma square in $\mathrm{Heis}(\mathcal{H})$)

$$\begin{tikzcd}
& ([2],[2]) \arrow[rr, "(m,m)"] \arrow[dd, "(\bar{c})"'] & & ([1],[1]) \arrow[dd, "(c)"] \\
([2]) \arrow[ur, "(\Delta(F),\mathrm{Id})"] \arrow[rr, "m"] \arrow[dd, "i_{\Delta(F)}"'] & & ([1]) \arrow[ur, "(F,\mathrm{Id})"] \arrow[dd, "i_F"] & \\
& ([4]) \arrow[rr, "\overline{m}"] & & ([2]) \\
([4]) \arrow[ur, equal] \arrow[rr, "\overline{m}"] & & ([2]) \arrow[ur, equal] &
\end{tikzcd}$$

Now, as we noted above, the back and top faces are comma squares in $\mathbf{Cat}_{ex}^{\otimes}$. The front face is therefore also comma, as their composition. This front face is exactly the square that we wanted to show is comma for the Heisenberg relation to hold.

Proof of Lemma 5.5. Suppose we have another square with the same bottom right corner, i.e.

$$\begin{tikzcd}
& ([2]) \arrow[rr, "m"] \arrow[dd, equal] & & ([1]) \arrow[dd, equal] \\
(C_i) \arrow[ur, "\varphi"] \arrow[rr] & & () \arrow[ur, "F"] \arrow[dd, equal] & \\
& ([2]) \arrow[rr, "m"] & & ([1]) \\
() \arrow[ur, "\Delta(F)"] \arrow[rr, equal] & & () \arrow[ur, "F"] &
\end{tikzcd}$$

First, the map φ must sit over a map which sends everything to the base point, in order for the whole square to sit over a commutative square. So we may assume that $(C_i) = ()$ and that φ is the selection of an element of [2]. In all, we can assume that the top square is of the form

$$\begin{tikzcd}
() \arrow[r, equal] \arrow[d, "X"'] & () \arrow[d, "F"] \arrow[dl, Rightarrow, "\gamma"'] \\
([2]) \arrow[r, "m"] & ([1])
\end{tikzcd}$$

i.e. $\gamma : F \to mX$. So we can form the following cube

$$\begin{tikzcd}
 & ([2]) \arrow[rr, "m"] \arrow[dd, equal] & & ([1]) \arrow[dd, equal] \\
() \arrow[ur, "X"] \arrow[rr, equal] \arrow[dd, equal] & & () \arrow[ur, "F"'] \arrow[dd, equal] & \\
 & ([2]) \arrow[rr, "m"] & & ([1]) \\
() \arrow[ur, "\Delta(F)"] \arrow[rr, equal] & & () \arrow[ur, "F"'] &
\end{tikzcd}$$

with left face given by $\Delta(F) \xrightarrow{\Delta(\gamma)} \Delta mX \xrightarrow{\epsilon} X$.

To finish our proof, we need to show

Claim. *This cube is commutative, and is unique as such.*

First, note that the left face we have specified is the image of γ under the canonical morphism

$$\mathrm{Hom}(F, mX) \xrightarrow{adj} \mathrm{Hom}(\Delta(F), X)$$

constructed from the adjunction data.

Now note that the composition on one side of the cube is $F \xrightarrow{\gamma} mX$, and on the other side it is $F \xrightarrow{\eta} m\Delta(F) \xrightarrow{m(adj(\gamma))} mX$, which is exactly $adj^{-1}(adj(\gamma)) = \gamma$.

From the above it is clear that any such commutative cube with left face $F \xrightarrow{\psi} mX$ must satisfy $adj^{-1}(\psi) = \gamma$ so $\psi = adj(\gamma)$ and it must be the same cube. $\square$

5.4 Fock space action in the case of the category $\mathcal{P}$

In this section we consider the example of the SSH category $\mathcal{P}$ of polynomial functors and see in detail the categorical Fock space action constructed from the SSH structure.

Consider the coCartesian arrow $c = (c_1, c_2) : (\{1\}, \{2\}) \xrightarrow{+} \{1, 2\}$ in $\mathbf{FinSet}^{\sqcup}$. Then, from the definition of the functor $\mathcal{H}$ giving the SSH structure on $\mathcal{P}$, for any $F \in \mathcal{P}$ the functor $i_F : \mathcal{P} \to \mathcal{P}^{\otimes[2]}$ is given by the formula:

$$i_F X(V) = F(c_0^* V) \otimes X(c_1^* V)$$

where $X \in \mathcal{P}, V \in \mathrm{Sh}(\{1, 2\})$.Its left and right adjoints are both given by the functor j_F:

$$j_F : \mathcal{P}^{\otimes[2]} \to \mathcal{P}, \qquad j_F \Phi(V) = \mathrm{Hom}_{\mathcal{P}}(F, \Phi(\boxtimes_c(-, V))$$

where $\boxtimes_c(W, V)$ is the sheaf defined by $c_0^* \boxtimes_c (W, V) = W, c_1^* \boxtimes_c (W, V) = V$. The functors m_F, Δ_F are defined by composing

$$m_F = m \circ i_F \qquad \Delta_F = j_F \circ \Delta$$

Note that in $\mathcal{P}$ these functors are biadjoint, so there is no need to distinguish between Δ_F^r and Δ_F^l and we just write Δ_F.

The isomorphisms

$$m_F m_G \cong m_{F\otimes G}$$
$$\Delta_G \Delta_F \cong \Delta_{F\otimes G}$$

come from the associator of the monoidal structure on the category **Vect**. As we saw in section §5.2, the non-trivial part of the isomorphism $\Delta_F m \cong m\Delta^2_{\Delta(F)}$ is given by the mate of the square (5.4). We want to check that the mate is invertible in the case of $\mathcal{P}$. Since we are now working with a concrete example, we will just compute the mate using the formula from Appendix B and check it is an invertible morphism.

Consider the arrow $\bar{c} : ([2], [2]) \to ([4])$ in $\mathbf{FinSet}^{\sqcup}$ given by the maps $\bar{c}_0 : {}^{1\mapsto 1}_{2\mapsto 3}, \bar{c}_1 : {}^{1\mapsto 2}_{2\mapsto 4}$ and let

$$i_\Phi : \mathcal{P}^{\otimes[2]} \to \mathcal{P}^{\otimes[4]} \qquad i_{\Delta\Phi}\Psi(V) = \Phi(\bar{c_0}^* V) \otimes \Psi(\bar{c_1}^* V)$$
$$j_\Phi : \mathcal{P}^{\otimes[4]} \to \mathcal{P}^{\otimes[2]}, \qquad j_\Phi \Omega(V) = \mathrm{Hom}(\Phi, \Omega(\boxtimes_{\bar{c}}(-, V))))$$

The functors i_Φ, j_Φ are biadjoint. Using the adjunction morphisms $\eta : \mathrm{Id} \to j_F i_F$ and $\epsilon : i_{\Delta(F)} j_{\Delta(F)} \to \mathrm{Id}$, the mate of the square (5.4) is given by:

$$m j_{\Delta(F)} \xrightarrow{\eta} j_F i_F m j_{\Delta(F)} \xrightarrow{\alpha} j_F \overline{m} i_{\Delta(F)} j_{\Delta(F)} \xrightarrow{\epsilon} j_F \overline{m} \tag{5.5}$$

The maps η and ϵ are given by the following formulas:

$$\begin{aligned} G(V) \xrightarrow{\eta} j_F i_F(G)(V) &= \mathrm{Hom}(F, F(-) \otimes G(V)) \\ &= \mathrm{End}(F) \otimes G(V) \\ x &\mapsto \mathrm{Id} \otimes x \end{aligned} \tag{5.6}$$

$$\begin{gathered} i_{\Delta F} j_{\Delta F}(\Omega)(V) = \Delta F(\bar{c}_0^* V) \otimes \mathrm{Hom}(\Delta F, \Omega(\boxtimes_{\bar{c}}(-, \bar{c}_1^* V))) \xrightarrow{\epsilon} \Omega(V) \\ x \otimes \tau \mapsto \tau_{\bar{c}_0^* V}(x) \end{gathered} \tag{5.7}$$

(note that for any sheaf $V \in \mathrm{Sh}([4])$ the sheaf $\boxtimes_{\bar{c}}(\bar{c}_0^* V, \bar{c}_1^* V)$ is canonically isomorphic to V)

Take $\Omega \in \mathcal{P}^{\otimes 4}, U \in \mathrm{Sh}([1])$ and let $a : [2] \to [1]$. Then

$$mj_{\Delta(F)}\Omega(U) = \mathrm{Hom}(\Delta F, \Omega(\boxtimes_{\bar{c}}(-, \bar{c}_1^* U))) \tag{5.8}$$

We want to take $x \in mj_{\Delta(F)}\Omega(U)$ and follow it through the map. Denote by W some vector space and v an element in $F(W)$, then

$$\begin{aligned}
\eta(x)_W &= (v \mapsto v \otimes x) \\
\alpha(\eta(x))_W &= (v \mapsto \eta_L(v) \otimes m(x)) \\
\epsilon(\alpha(\eta(x)))_W &= (v \mapsto m(x)(\eta_L(v)))
\end{aligned}$$

where η_L is the map $F \to m\Delta F$ from the adjunction (m, Δ) (see Appendix C for details) and $m(x)$ is the application of m to x, which is a map $m\Delta F$ to $m\Omega(\boxtimes_{\bar{c}}(-, \bar{c}_1^* U))$. Altogether, we have computed that the mate takes the map x to $m(x)$ precomposed with the unit η_L, but this is exactly the construction of the isomorphism

$$\mathrm{Hom}(\Delta F, \Omega(\boxtimes_{\bar{c}}(-, \bar{c}_1^* U))) \xrightarrow{\sim} \mathrm{Hom}(A, m\Omega(\boxtimes_{\bar{c}}(-, \bar{c}_1^* U)))$$

for the adjoint pair m, Δ, so it is invertible.

6 Equivariant polynomial functors

In this section we extend our example of polynomial functors to a class of examples which we call G-equivariant polynomial functors. This is the direct categorical analog of the example of representations of wreath products $S_n[G]$ considered by Zelevinsky in [Zel81].

6.1 Definition

Recall that we defined the SSH structure on $\mathcal{P}$ by giving a symmetric monoidal functor $\mathbf{FinSet}^{\sqcup} \xrightarrow{\mathcal{H}} \mathbf{Cat}_{ex}^{\otimes}$. On objects it was defined by

$$\mathcal{H}(S) = \mathcal{P}^{\otimes S}$$

where $\mathcal{P}^{\otimes S}$ was defined to be the category of polynomial functors from $\mathrm{Sh}(S)$ to **Vect**.

Let G be a finite group. We define a functor $\mathcal{H}_G$ in the same way we defined $\mathcal{H}$, but replacing $\mathrm{Sh}(S)$ with $\mathrm{Sh}_G(S)$, i.e. sheaves of G-representations on S.

Remark 6.1. More naturally, the assignment $S \mapsto \mathrm{Sh}_G(S)$ can be split into two pieces:

1. The imbedding $\mathbf{FinSet} \hookrightarrow G\,\mathrm{Set}$ given by $S \mapsto S$ with the trivial G action.
2. The functor $G\,\mathrm{Set} \to \mathbf{Vect\,Cat}$ which sends a G-set X to the category of sheaves on the groupoid X/G.

It might be interesting to consider on its own the functor $G\,\mathrm{Set} \to \mathbf{Vect\,Cat}$.

Define $\mathcal{P}_G$ to be $\mathcal{H}_G([1])$.

6.2 A manifestation of Zelevinsky's decomposition Theorem

Zelevinsky's main theorem about PSH algebras, is that they are all tensor products of many copies of Λ. His proof is somewhat combinatorial, and gives the morphism to the tensor product only up to a non canonical choice.

In this section we would like to show a categorical analog of it, where we obtain a canonical equivalence (of SSH categories) to a tensor power of $\mathcal{P}$. We conjecture that a similar result holds in general for SSH categories.

Consider the functor $Y : \mathrm{Sh}_G(S) \to \mathrm{Sh}(S \times \mathrm{Irr}\, G)$ given by

$$Y(V)(s,\rho) = \mathrm{Hom}(\rho, Y_s)$$

Since we are working in characteristic zero, this functor is an equivalence. Hence it induces an equivalence

$$\begin{aligned}\mathcal{P}^{\otimes(S\times \mathrm{Irr}\, G)} &= \mathrm{PolyFun}(\mathrm{Sh}(S\times \mathrm{Irr}\, G), \mathbf{Vect}) \\ &\xrightarrow{\sim} \mathrm{PolyFun}(\mathrm{Sh}_G(S), \mathbf{Vect}) = \mathcal{H}_G(S)\end{aligned}$$

This equivalence obviously commutes with the sheaf operations on the S component, and hence induces an equivalence of SSH structures between $S \mapsto \mathcal{P}^{\otimes(S\times \mathrm{Irr}\, G)}$ (as in 2.18) and $\mathcal{H}_G$.

In particular we have that $\mathcal{P}^{\otimes\, \mathrm{Irr}\, G}$ is equivalent to $\mathcal{P}_G$ as a symmetric monoidal category, since the symmetric monoidal structure on each of them comes from the SSH structure that they are a part of.

Remark 6.2. The functor Y has an inverse, which we denote Y^{-1} given by

$$Y^{-1}(V)(s) = \bigoplus_{\rho\in \mathrm{Irr}\, G} V_{(s,\rho)} \otimes \rho$$

This will be of use to us in the next subsection.

Note that this functor is defined up to a choice of concrete models for the irreducible representations of G.

6.3 Connection with wreath products

In [Zel81], Zelevinsky considers the PSH algebra $\Lambda_G := K\left(\bigoplus_n \mathrm{Rep}(S_n[G])\right)$ where $S_n[G]$ is the wreath product $G^n \rtimes S_n$, multiplication is given by induction and inner product is given by dimension of hom-space.

Proposition 6.3. *There is a natural map of PSH algebras $\Lambda_G \to K(\mathcal{P}_G)$, and this map is an isomorphism.*

Proof. We will first construct a contravariant exact functor

$$L : \bigoplus_n \mathrm{Rep}(S_n[G]) \to \mathcal{P}_G$$

Denote $G_n := S_n[G]$, and let ρ be a representation of G_n. We define $L_\rho \in \mathcal{P}_G$ as follows: Take $V \in \mathrm{Sh}_G([1]) = \mathrm{Rep}(G)$. Then $V^{\otimes n}$ is naturally a representation of G_n, and we define

$$L_\rho(V) := \mathrm{Hom}_{G_n}(\rho, V^{\otimes n})$$

Since we are in characteristic 0, this functor is obviously exact, so induces a map of K-groups, and we need to check that it preserves all the relevant structures.

First, it preserves multiplication because

$$\begin{aligned} m(L_{\rho_1}, L_{\rho_2})(V) &= L_{\rho_1}(V) \otimes L_{\rho_2}(V) \\ &= \mathrm{Hom}_{G_{n_1}}(\rho_1, V^{\otimes n_1}) \otimes \mathrm{Hom}_{G_{n_2}}(\rho_2, V^{\otimes n_2}) \\ &\cong \mathrm{Hom}_{G_{n_1} \times G_{n_2}}(\rho_1 \boxtimes \rho_2, V^{\otimes n_1} \boxtimes V^{\otimes n_2}) \\ &\cong \mathrm{Hom}_{G_{n_1+n_2}}(\rho_1 \cdot \rho_2, V^{\otimes (n_1+n_2)}) \\ &= L_{\rho_1 \cdot \rho_2}(V) \end{aligned}$$

Secondly, it is fully faithful, since by evaluating on $\Bbbk[G]$ we see that $\mathrm{Hom}(L_{\rho_1}, L_{\rho_2})$ is canonically isomorphic to $\mathrm{Hom}(\rho_2, \rho_1)$ with the isomorphism given by the map $\mathrm{Hom}(\rho_2, \rho_1) \to \mathrm{Hom}(L_{\rho_1}, L_{\rho_2})$ induced by L. The upshot is that the map of algebras preserves the inner product, and in all is a map of PSH algebras.

Now, compose the functor L with the equivalence induced by Y^{-1} (from Remark 6.2). Since both functors preserve multiplication, we get a map of algebras $\Lambda_G \to K(\mathcal{P}^{\otimes \operatorname{Irr} G})$.

If $\rho \in \operatorname{Irr} G$, this composition of functors takes ρ to a polynomial functor

$$R_\rho : \mathrm{Sh}(\operatorname{Irr} G) \to \mathbf{Vect}$$

given by

$$R_\rho(V) = V_\rho$$

It is obvious that R_ρ defines a primitive irreducible element of $K(\mathcal{P}^{\otimes \operatorname{Irr} G})$. Moreover, these are all of the irreducible primitives in this algebra.

Also, as noted in [Zel81], $\operatorname{Irr} G$ is the set of all irreducible primitives in Λ_G. In all, we have a map of PSH algebras, which sends the set of irreducible primitives on the LHS bijectively onto the set of irreducible primitives on the RHS, so it must be an isomorphism.

As a consequence, the map induced by L is also an isomorphism.

$\square$

A Commutative cubes

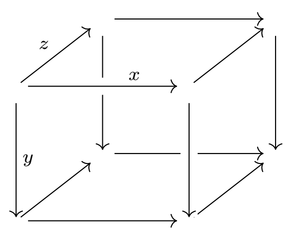
In this section we explain what we mean by the term "commutative cube" in a 2-category. The general statement is as follows: for an n-dimensional cube given an ordering of the coordinates there is a concise way to orient the 2-morphisms, thus getting a diagram of 2-morphisms for any cube. We will say that the cube is commutative if this diagram is. The ordering used in this article is described below. A good general reference for this topic is [Gra76].

A.1 2-cubes

Consider a square, where we have ordered the coordinates as $x < y$. Then we orient the 2-morphism by the lexicographical order, i.e. $xy \to yx$.

A.2 3-cubes

Order the three coordinates in the cube as $x < y < z$. The edges of the cube are all oriented positively in one of these directions, i.e. as in the diagram:



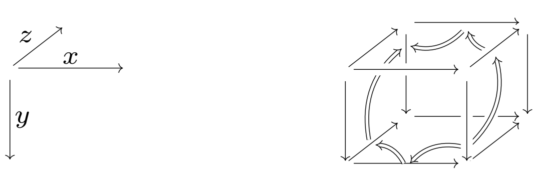
The 2-morphisms are oriented as in the previous section. E.g. we have a morphism $xy \to yx$, and via whiskering we get a morphism $xyz \to yxz$.

In this way we get a diagram of all 2-morphisms between full paths on the cube, which is a hexagon

$$\begin{tikzcd}
 & yxz \arrow[r] & yzx \arrow[dr] & \\
xyz \arrow[ur] \arrow[dr] & & & zyx \\
 & xzy \arrow[r] & zxy \arrow[ur] &
\end{tikzcd} \tag{A.1}$$

We say that the cube is commutative if this diagram commutes.

We can draw this diagram in the cube in the following way:



A.3 Higher dimensional cubes

In the same way we can orient the 2-morphisms on any k-cube by ordering the coordinates as $x_1, \dots, x_k$.

For instance for a 4-cube the resulting diagram is a 3-dimensional shape with 8 faces which are hexagons and 6 faces which are squares (a *truncated octahedron*). The entire diagram commutes iff each face commutes. The hexagons correspond to the sub 3-cubes, and the squares are related to the four-interchange law in a 2-category. So we see that a 4-cube in a 2-category commutes iff every sub 3-cube in it commutes.

More generally we have

Theorem 4 (Gray, [Gra76])**.** *A k-cube in a 2-category commutes iff every sub 3-cube in it commutes.*

B The Beck-Chevalley condition for squares and cubes

Let $\mathcal{C}$ be a 2-category, and consider a square (sometimes called a *quintet*)

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow\alpha"] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd} \tag{B.1}$$

i.e. a 2-morphism $\alpha : h \circ g \to i \circ f$ (which is not necessarily an isomorphism).

Suppose that the verticals h, f both have right adjoints h_R, f_R, with given unit-counit pairs, then we can form the square (called the *right mate* of the above square)

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, phantom, "\Searrow\alpha_R"] & B \\ C \arrow[r, "i"] \arrow[u, dashed, "f_R"] & D \arrow[u, dashed, "h_R"'] \end{tikzcd}$$

where α_R is the composition

$$g \circ f_R \to h_R \circ h \circ g \circ f_R \xrightarrow{\alpha} h_R \circ i \circ f \circ f_R \to h_R \circ i$$

Definition B.1. A square as in (B.1) with 2-morphism α is said to satisfy the right *Beck-Chevalley condition* if α_R is invertible.

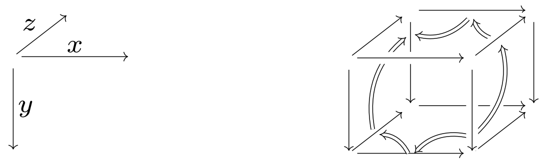
Similarly, if the horizontals g, i both have left adjoints, we can define the left Beck-Chevalley condition via the left mate square

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow\alpha_L"] & B \arrow[l, dashed, "g_L"'] \arrow[d, "h"] \\ C & D \arrow[l, dashed, "i_L"] \end{tikzcd}$$

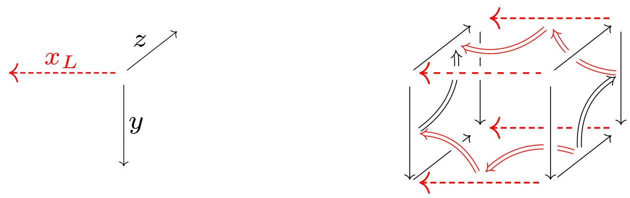
Remark B.2. Note that a square satisfies the right BC condition iff it satisfies the left BC condition, when both are defined, hence we can omit the words *left* or *right*.

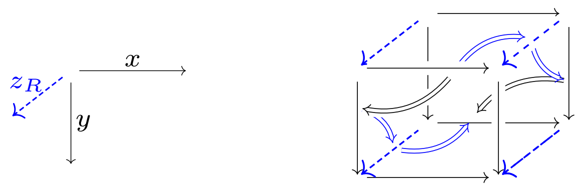
To get relations between mates of squares we need to look at cubes.

Lemma B.3. *Consider a commutative cube (see Appendix A for details)*



and suppose that all arrows in the x (resp. z) direction have left (resp. right) adjoints, with given unit/counit. Then the left (right) mate of the cube, i.e. the cube obtained from taking the left (right) mates of all faces involving direction x (z) arrows, commutes. Explicitly, they are the following cubes





Using the description from Appendix A the first cube corresponds to the ordering of coordinates $y < z < x$ and the second to the ordering $z < x < y$.

Proof. A direct computation, using the 4-interchange law in a 2-category, and the fact that the original cube commutes. ☐

C Adjunction of inverse and direct image for sheaves on finite sets

Let $\varphi : S \to T$ be a map of sets, and let φ^*, φ_* be the functors of inverse and direct image between the categories of sheaves. Let $V \in \mathrm{Sh}(S), W \in \mathrm{Sh}(T), A \subset S, B \subset T$. Note that for a sheaf on a finite set we have

$$V(A) = \prod_{a \in A} V_a$$

so we have the following formulas:

$$\varphi^* W(A) = \prod_{a \in A} W_{\varphi(a)} \tag{C.1}$$

$$\varphi_* V(B) = V(\varphi^{-1}(B)) \tag{C.2}$$

and so

$$\varphi^* \varphi_* V(A) = \prod_{a \in A} V(\varphi^{-1}(\varphi(a))) \tag{C.3}$$

$$\varphi_* \varphi^* W(B) = \prod_{a, \varphi(a) \in B} W_{\varphi(a)} \tag{C.4}$$

The unit and counit of the adjunction $\varphi^* \dashv \varphi_*$ are given by maps

$$\epsilon_R : \varphi^* \varphi_* V(A) = \prod_{a \in A} V(\varphi^{-1}(\varphi(a))) \to \prod_{a \in A} V_a = V(A) \tag{C.5}$$

$$\eta_R : W(B) = \prod_{b \in B} W_b \to \prod_{a, \varphi(a) \in B} W_{\varphi(a)} = \varphi_* \varphi^* W(B) \tag{C.6}$$

where ϵ_R is given by restrictions and η_R by the diagonal maps.

There is also an adjunction $\varphi_* \dashv \varphi^*$ given by maps

$$\eta_L : \varphi^* \varphi_* V(A) = \prod_{a \in A} V(\varphi^{-1}(\varphi(a))) \leftarrow \prod_{a \in A} V_a = V(A) \tag{C.7}$$

$$\epsilon_L : W(B) = \prod_{b \in B} W_b \leftarrow \prod_{a, \varphi(a) \in B} W_{\varphi(a)} = \varphi_* \varphi^* W(B) \tag{C.8}$$

where η_L is given by extension by 0, and ϵ_L is given by the sum maps.

D Limits

Let K be a small category and $\mathcal{C}$ an $(\infty, 2)$-category.

Definition D.1. We say that $x \in \mathcal{C}$ is *final* if the canonical map $\mathcal{C}_{/x} \to \mathcal{C}$ has weakly contractible fibers.

D.1 Regular limits

Let $F : K \to \mathcal{C}$ be a functor. The limit of F can be thought of as follows (cf. [GP99]):

Denote also by F the functor $\mathrm{pt} \to \mathcal{C}^K$ which sends pt to F, and by *diag* the diagonal embedding $\mathcal{C} \to \mathcal{C}^K$. Then we may form the comma category

$$\begin{tikzcd} (diag \downarrow F) \arrow[r, "p"] \arrow[d] & \mathcal{C} \arrow[d, "diag"] \\ \mathrm{pt} \arrow[r, "F"] & \mathcal{C}^K \end{tikzcd}$$

A final object L in $(diag \downarrow F)$ is exactly a limit of F together with all the relevant morphisms to the diagram. If we want just the object, then it is given by $p(L)$.

D.2 Generalized limits and comma squares

In the above, we only dealt with conical limits, i.e. we used the map $\mathcal{C} \xrightarrow{diag} \mathcal{C}^K$ which comes from the map $K \to \mathrm{pt}$.

However, we can replace it by any map $K \xrightarrow{W} L$. In such cases we call the resulting object a W-limit (this is closely related to the notion of *weighted* limit appearing in the literature).

Example D.2 (Comma Squares). Let $K \to L$ be the map

$$
\begin{tikzcd}
 & \bullet \arrow[d] & & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
\bullet \arrow[r] & \bullet & \arrow[u, phantom, "\overset{W}{\hookrightarrow}"] & \bullet \arrow[r] & \bullet
\end{tikzcd}
$$

and let $\mathcal{C}$ be a 2-category.

A map $K \xrightarrow{F} \mathcal{C}$ is a "bottom right corner", and a W-limit of F is exactly a comma square with this bottom right corner.

Let us analyze this in more detail:

Let $F = \begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$ be a bottom right corner in $\mathcal{C}$, and consider the square:

$$
\begin{tikzcd}
(W^* \downarrow F) \arrow[r, "p"] \arrow[d] & \mathcal{C}^{\square} \arrow[d, "W^*"] \\
\mathrm{pt} \arrow[r, "F"] & \mathcal{C}^{\lrcorner}
\end{tikzcd}
$$

Objects of $(W^* \downarrow F)$ are pairs $(S \in \mathcal{C}^{\square}, \alpha : W^*S \to F)$. So they can be thought of as diagrams:

$$
\begin{tikzcd}
 & Z \arrow[rr] \arrow[dd] & & W \arrow[dd] \\
X \arrow[ur] \arrow[rr] & & Y \arrow[ur] \arrow[dd] & \\
 & B \arrow[rr] & & C \\
 & & A \arrow[ur] &
\end{tikzcd}
$$

A morphism in $(W^* \downarrow F)$ is then a commutative diagram

$$\begin{tikzcd}
& Z' \arrow[rr] \arrow[dd] & & W' \arrow[dd] \\
X' \arrow[ur] \arrow[rr] \arrow[dd] & & Y' \arrow[ur] \arrow[dd] & \\
& Z \arrow[rr] \arrow[dd] & & W \arrow[dd] \\
X \arrow[ur] \arrow[rr] & & Y \arrow[ur] \arrow[dd] & \\
& B \arrow[rr] & & C \\
& & A \arrow[ur] &
\end{tikzcd}$$

Such that we have

$$\begin{tikzcd}
Z' \arrow[rr] \arrow[dd] & & W' \arrow[dd] \\
& Y' \arrow[ur] \arrow[dd] & \\
Z \arrow[rr] \arrow[dd] & & W \arrow[dd] \\
& Y \arrow[ur] \arrow[dd] & \\
B \arrow[rr] & & C \\
& A \arrow[ur] &
\end{tikzcd}
=
\begin{tikzcd}
Z' \arrow[rr] \arrow[dd] & & W' \arrow[dd] \\
& Y' \arrow[ur] \arrow[dd] & \\
B \arrow[rr] & & C \\
& A \arrow[ur] &
\end{tikzcd}$$

Suppose now that we an object in $(W^* \downarrow F)$

$$L = \begin{tikzcd}
& Z \arrow[rr] \arrow[dd] & & W \arrow[dd] \\
X \arrow[ur] \arrow[rr] & & Y \arrow[ur] \arrow[dd] & \\
& B \arrow[rr] & & C \\
& & A \arrow[ur] &
\end{tikzcd}$$

By composition we can form a new object

$$\tilde{L} = \begin{tikzcd}
& B \arrow[rr] \arrow[dd, equal] & & C \arrow[dd, equal] \\
X \arrow[ur] \arrow[rr] & & A \arrow[ur] \arrow[dd, equal] & \\
& B \arrow[rr] & & C \\
& & A \arrow[ur] &
\end{tikzcd}$$

Define $(W^* \downarrow F)^o$ to be the subcategory of $(W^* \downarrow F)$ with $\alpha = \mathrm{Id}$ (i.e. in the form of $\tilde{L}$). Then we have

Lemma D.3. *$\tilde{L}$ is final in $(W^* \downarrow F)^o$ iff L is final in $(W^* \downarrow F)$.*

Proof. Consider the category $(\widetilde{W^* \downarrow F})$ with objects - commutative cubes

$$\begin{tikzcd}
& Z \arrow[rr] \arrow[dd] & & W \arrow[dd] \\
X \arrow[ur] \arrow[rr] \arrow[dd, equal] & & Y \arrow[ur] \arrow[dd] & \\
& B \arrow[rr] & & C \\
X \arrow[ur] \arrow[rr] & & A \arrow[ur] &
\end{tikzcd}$$

with front and left faces invertible.

We have maps

$$\begin{tikzcd}
& (\widetilde{W^* \downarrow F}) \arrow[dl, "p"'] \arrow[dr, "q"] & \\
(W^* \downarrow F) & & (W^* \downarrow F)^o
\end{tikzcd}$$

given by the obvious projections. Note that p is a trivial fibration and q is a left fibration. In both cases an object in $(\widetilde{W^* \downarrow F})$ iff its image is final, and both maps are essentially surjective. The result follows. □

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