Partial Groups and Homology

Groups, Partial Groups, Homology, Topology

- The homology of a group.
- The bar construction.
- Can the homology of a group be computed effectively?
- Partial Groups.
- Examples from Topology.

How is the homology of a group G defined?

G acts properly discontinuously on the infinite simplex Δ with vertices labeled by G giving a covering space

$$\Delta \Rightarrow BG.$$

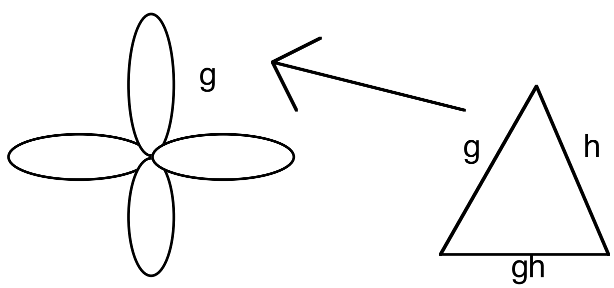
$$\pi_1(BG) = G; \quad \pi_k(BG) = 0, \ \kappa > 1.$$

The base space BG is a K(G,1).

$H_\star(BG)$ is the classical homology of the discrete group G.

The simplicial structure of BG

BG has one vertex; a 1-simplex attached for each element of G; a 2-simplex attached for each pair of elements, etc.



The resulting space is a C.W. complex and the associated chain complex $C_\star(BG)$ is the bar construction.

The Nerve of G

The construction, formulated simplicially, is the Nerve of G.

$$G_0=\{1\}, \quad G_m= G^m,$$

$$\partial_0(g_0,\ldots,g_m) = (g_1,\ldots,g_m)$$

$$\partial_m(g_0,\ldots,g_m) = (g_0,\ldots,g_{m-1})$$

$$\partial_i\,(g_0,\ldots,g_m) = (g_0,\ldots,g_ig_{i+1},\ldots g_m)$$

$$\eta_i\,(g_0,\ldots,g_m) = (g_0,\ldots, g_i,1,g_{i+1},\ldots g_m).$$

Theorem of Kan and Thurston

Every connected space has the homology of a $K(G,1)$

Topology (15), 1976.

The Bar Construction, explicitly

$C_k(BG)$ = free abelian group on $(g_1,\ldots,g_k)$

$d = \Sigma\, d_i : C_k(BG) \rightarrow C_{k-1}(BG)$ is the boundary,

where the face maps d_i are given by

$$d_0(g_1,\ldots,g_k) = (g_2,\ldots,g_k),$$
$$d_i(g_1,\ldots,g_k) = (g_1,\ldots, g_i \circ g_{i+1},\ldots,g_k),$$
$$d_k(g_1,\ldots,g_k) = (g_1,\ldots,g_{k-1}).$$

Computing Homology

The circle is a $K(Z,1)$ with one 0-cell and one 1-cell.

The bar construction uses a simplicial model with an infinite number of cells in each positive dimension.

Is there a natural, but more efficient way to build a $K(G,1)$?

Partial Groups

A set P is a partial group if associated to each pair $(x,y) \in P \times P$ there is at most one product element $x \circ y$ so that

1. there exists an element $1 \in P$ satisfying $x \circ 1 = 1 \circ x = x$ for each $x \in P$,

2. for each $x \in P$ there exists an element $x^{-1} \in P$ so that $x \circ x^{-1} = x^{-1} \circ x = 1$,

3. if $x \circ y = z$ is defined then so is $y^{-1} \circ x^{-1} = z^{-1}$.

Associativity for Partial Groups

Let P^k be the k-fold product of P, and $\Lambda = \cup_k P^k$.
Let an arrow $\rightarrow$ denote the transitive relation on Λ generated by $(x_1,\ldots,x_k) \in P^k$ is related to $(x_1,\ldots,x_i \circ x_{i+1},\ldots,x_k) \in P^{k-1}$ whenever $x_i \circ x_{i+1}$ is defined.

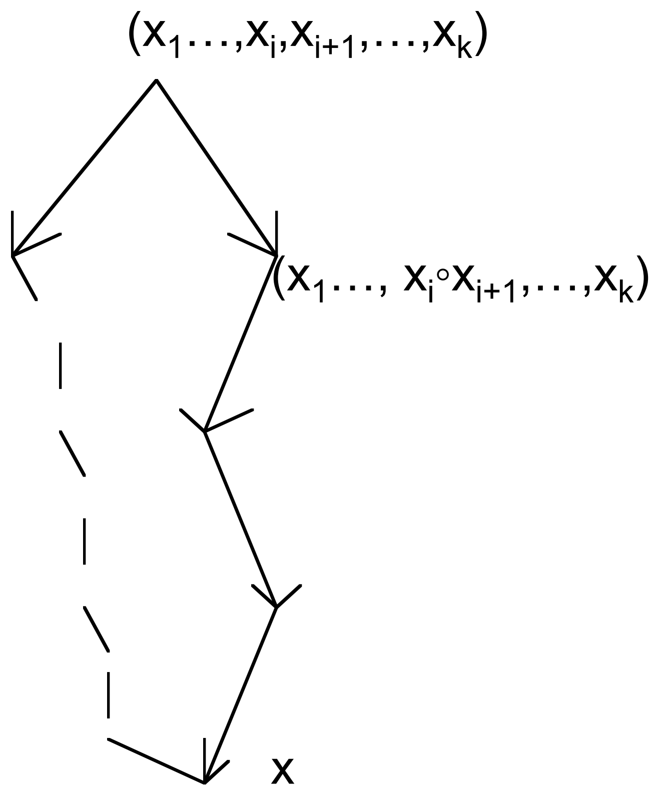
A partial group P is associative if the following holds.

Let $x,s,t \in \Lambda$. If $x \rightarrow s$, $s \in P$, and $x \rightarrow t$ where t is minimal with respect to $\rightarrow$, then $s = t$.

Informal Definition of Associativity

The associativity condition says that if some sequence of multiplications leads from a given k-tuple of elements to a single element, then no other way of multiplying the entries can terminate until a single, and necessarily unique, element is attained.

Schematically



The structure of partial groups

- The nerve of a partial group P, constructed with respect to its composition, is a well defined simplicial set P_* with a classifying space BP.
- If P is associative then BP is an Eilenberg-Maclane space of type (G, 1), where G is the universal group of the partial group, that is, the free group on the elements of P, modulo the relations $x \cdot y = x \circ y$ whenever $x \circ y$ is defined.
- In an associative partial group, every generator represents a non-trivial element in the universal group. Moreover, any minimal word, defined to be one in which no two successive elements are composable, represents a non-trivial element of the universal group.
- The word problem is solvable in G(P) in the sense that x=1 in G(P) if and only if $x \rightarrow 1$.

Pregroups

A pregroup consists of a set P containing an element 1, each element $s \in S$ has a unique inverse s^{-1} and to each pair of elements $s, t \in S$ there is defined at most one product $st \in S$ so that

(a) $1s = s1 = s$ is always defined.

(b) $ss^{-1} = s^{-1}s = 1$ is always defined.

(c) If st is defined then $t^{-1}s^{-1}$ is defined and equal to $(st)^{-1}$.

(d) If rs and st are defined then $r(st)$ is defined if and only if $(rs)t$ is defined, in which case the two are equal.

(e) If qr, rs and st are defined then either $q(rs)$ is defined or $r(st)$ is defined.

Partial Groups vs Pregroups

Every pregroup is a partial group.

The converse is not true.

Original references:

Pregroups: Stallings, The Cohomology of Pregroups, Springer Lecture Notes 319, 1973.

Partial Groups: Jekel, Simplicial K(G,1)'s, manuscripta mathematica (21), 1977.

Two General Examples

- A union of groups with a common identity element has a partial group structure.
- Consider the fundamental groupoid of a space, and identify all the points (objects) of the groupoid to a single point. Define composition between paths (morphisms) to be defined whenever the composition is defined in the groupoid.The resulting set with this composition is a partial group.

“Free” Partial Groups

A free partial group is a partial group P where the only compositions are the trivial ones.

If P has no element (other than 1) which is its own inverse then the universal group of P is free.

To find a basis write P- {1} as $P^+ \cup P^-$ where $x \in P^+$ iff $x^{-1} \in P^-$. A basis is the set P^+.

The C.W. complex which is its realization is a wedge of circles, but there are degenerate simplices like $(1,\ldots,x,x^{-1},\ldots,1)$.

Z_2 and the Infinite Dihedral Group

Z_2 is the universal group of the pregroup on a single letter which is its own inverse. So Z_2 is free in the sense that all compositions are trivial.

The infinite dihedral group is the subgroup of the affine group generated by 2-x and -x

The pregroup whose universal group is the infinite dihedral group consists of two letters, each of which is its own inverse, and no non-trival compositions.

The infinite dihedral group is just Z_2 v Z_2 .

Generally a dihedral group is a group generated by a pair of elements of order 2. The finite dihedral groups have non-trivial compositions.

The Discrete Euler Class

Let G be the discrete group of orientation preserving homeomorphisms of the circle $S^1 = R \cup \infty$.

There is an invariant in $H^2(BG)$, called the Discrete Euler Class.
It is the obstruction to reducing a circle bundle with structrure group G as a topological group to G as a discrete group.

Suppose $f \in G$ maps ∞ to b and a to ∞.
Then f determines two homeomorphisms:
a left branch, $f_L : (-\infty, a) \to (b,\infty)$, and
a right branch $f_R : (a,\infty) \to (-\infty, b)$.
If $\infty = a$, or $\infty = b$, then $f_L = f_R$.

f considered as a pair of local homeomorphisms of R has two asymptotes;
x=a and y =b.

Examples

-1/x as a function on R - {0} has two branches:

$(-1/x)_L : (-\infty, 0) \to (0,\infty)$ and $(-1/x)_R : (0, \infty) \to (\infty,0)$.

The branches determine a homeomorphism of the circle.

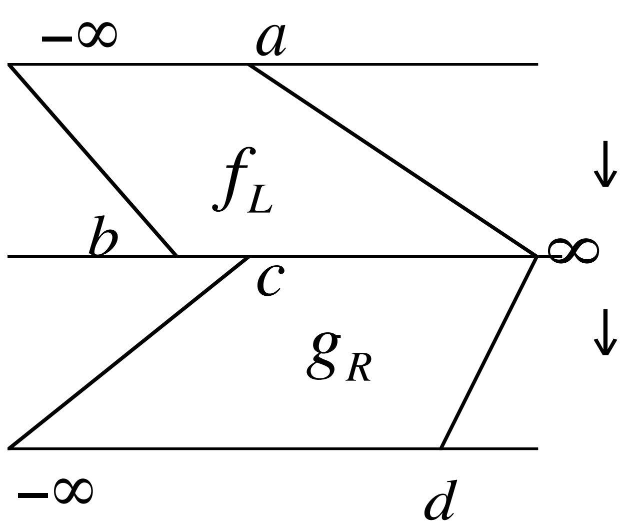
More generally a linear fractional transformation ax+b/cx+d
determines two branches (one if cx+d is never zero),
And extends to a homeomorphism of the circle.

Composition of branches

Suppose f , g $\in$ G determine the branches

$f_L : (-\infty, a) \rightarrow (b,\infty)$, and $f_R : (a,\infty) \rightarrow (-\infty, b)$

$g_L : (-\infty, c) \rightarrow (d,\infty)$, and $g_R : (c,\infty) \rightarrow (-\infty, d)$.



$G\_ = G_L \cup G_R$ is a partial group

The figure illustrates $g_R \circ f_L = (g \circ f)_R$ if $b \leq c$.

The composition $g\_ \circ f\_$ of two branches is defined if and only if the range of $f\_$ intersects the domain of $g\_$.

This partially defined multiplication makes the set of all branches $G_L \cup G_R$ of G into an associative partial group.

Reference: Jekel, The Euler class in homological algebra, Journal of Pure and Applied Algebra 215 (2011)

Structure of G_

It is proved there that

- The universal group Ĝ of G_ maps onto G.
- The kernel is isomorphic to Z.
- The characteristic cohomology class E(G) of the extension $0 \to Z \to \hat{G} \to G \to 1$ is the Discrete Euler Class of G.

Behavior of the Powers of E

- What can be said about the powers $E^p(G)$?

  $E^p(G)$ is non-zero for all p.

- Formulate $E^p(G)$ as an extension.

- Formulate $E^p(H)$ where H is a subgroup of G.

- Formulate $E^p(H)$ in such a way that useful conditions on H can be found for determining when $E^p(H)$ vanishes.

The Based Mapping Class Groups

The Based Mapping Class Group of a surface of genus g, $M_{g,*}$ **,** can be identified with a subgroup of G and so the Discrete Euler Class is an invariant.

The Discrete Euler Class of the groups $M_{g,*}$ exhibits the following behavior.

$E^n(M_{g,*})$ determines a non-zero element in $\mathrm{Hom}(H_{2n}(M_{g,*}), Z)$ for $n < g$,

$E^n(M_{g,*})$ is zero for $n > g$,

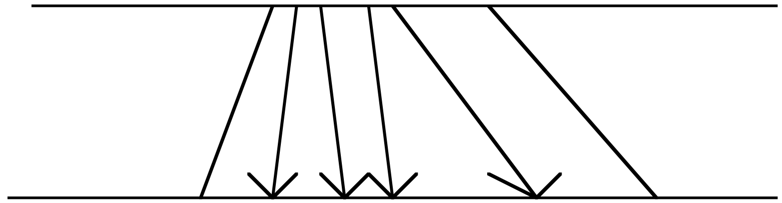
$E^g(M_{g,*})$ has torsion $2g(2g + 1)$ in $H^{2n}(M_{g,*})$

Reference: Jekel, Powers of the Euler Class, Advances in Mathematics 229 (2012)

Codimension-1 C^ω foliations

The classifying space for codimension-1, C^ω foliations is homotopy equivalent to BP where P is the partial group of maximally extended, local, orientation preserving, real analytic homeomorphism of the real line.

P is an associative partial group under composition, but is not a pregroup.



Reference: Jekel, On two theorems of A. Haefliger concerning foliations, Topology 15(1976)

Properties of G(P)

G(P) is uncountable and every generator represents a distinct element of infinite order.

All elements of P are conjugate in G(P).

$H_1\,(BP) = 0$.

What is $H_2\,(BP)$?

Questions

Is there a theory for Partial Groups which can shed light on the computation of homology?

Is it meaningful that the word problem is solvable in the universal group of an associative partial group?

Is it significant that the invariants for surfaces described here arise naturally from partial group structures?

Additional Notes

The following page is taken from the introduction to the preprint “Real Analytic Γ- Structures on Surfaces”, and gives some history and background on the problem of computing H_2 (BP).

Introduction to Real Analytic Γ- Structures on Surfaces

Solomon Jekel and Alberto Verjovsky

The modern theory of foliations began with dramatically contrasting results. In 1944 G. Reeb constructed a smooth codimension-one foliation on the 3-sphere, observing that it was not real analytic. Then in 1958 A. Haefliger proved that no codimension-one real analytic foliation can exist on any simply connected manifold. But despite great advances in foliation theory, especially during a period of intense activity in the 1970's, the relationship between smooth and real analytic foliations remains largely a mystery.To classify foliations, Haefliger broadened the definition of "foliation on a mani-fold" to "Γ- structure on a topological space". Each Γ has a classifying space BΓ, and questions about foliations, in particular those involving different degrees of differentiability, can often be formulated in terms of their algebraic invariants. For example,the non-existence of codimension-one real analytic foliations on simply connected manifolds is a consequence of the non-vanishing of the fundamental group of the classifying space $B\Gamma^\omega$. In contrast to the real analytic case, the smooth classifying space $B\Gamma^\infty$ is simply connected. So it is natural to ask how the homology groups and the homotopy groups of $B\Gamma^\omega$ compare to those $B\Gamma^\infty$. These two spaces are vastly different homotopically, but homologically they have surprising similarities. $B\Gamma^\infty$, is simply connected and, by results of Mather, π_2 and hence H_2 is zero as well. On the other hand, Haefliger proved that $B\Gamma^\omega$ is a $K(\pi, 1)$, with π an uncountable perfect group. Thurston subsequently constructed uncountably many non-cobordant, real analytic, codimension-one foliations on 3-manifolds showing that H^3 of both $B\Gamma^\omega$ and $B\Gamma^\infty$ have homomorphisms onto the reals in integer homology. At the end of the 1970's no homology of the two spaces was known to be different,and this observation led to speculation that the spaces were homology equivalent. We can now show that this is false: $H_2(B\Gamma^\omega, Z)$ is uncountable.