Cluster Structures of Double Bott-Samelson Cells

Daping Weng

Michigan State University

April 2019

Joint work with Linhui Shen

arXiv:1904.07992

Motivation: Bott-Samelson Variety

- Let G, B, W be defined as usual. Let $\mathbf{i} = (i_1, \ldots, i_l)$ be a reduced word of w. The Bott-Samelson variety associated to the reduced word $\mathbf{i}$ is

$$\mathrm{P}_{i_1} \underset{\mathrm{B}}{\times} \mathrm{P}_{i_2} \underset{\mathrm{B}}{\times} \ldots \underset{\mathrm{B}}{\times} \mathrm{P}_{i_l} \Big/ \mathrm{B}$$

where $\mathrm{P}_i = \mathrm{B} \sqcup \mathrm{B}s_i\mathrm{B}$.

Motivation: Bott-Samelson Variety

- Let G, B, W be defined as usual. Let $\mathbf{i} = (i_1, \ldots, i_l)$ be a reduced word of w. The Bott-Samelson variety associated to the reduced word $\mathbf{i}$ is

$$\mathrm{P}_{i_1} \underset{\mathrm{B}}{\times} \mathrm{P}_{i_2} \underset{\mathrm{B}}{\times} \ldots \underset{\mathrm{B}}{\times} \mathrm{P}_{i_l} \Big/ \mathrm{B}$$

  where $\mathrm{P}_i = \mathrm{B} \sqcup \mathrm{B}s_i\mathrm{B}$.

- Note that

$$\mathrm{P}_{i_1} \underset{\mathrm{B}}{\times} \ldots \underset{\mathrm{B}}{\times} \mathrm{P}_{i_l} = \bigsqcup_{\mathbf{j} \subset \mathbf{i}} (\mathrm{B}s_{j_1}\mathrm{B}) \underset{\mathrm{B}}{\times} \ldots \underset{\mathrm{B}}{\times} (\mathrm{B}s_{j_m}\mathrm{B})$$

  where $\mathbf{j} = (j_1, \ldots, j_m)$ runs over all subwords of $\mathbf{i}$ (not necessarily reduced). These can be thought of as “Bott-Samelson cell”.

Motivation: Bott-Samelson Variety

- Let G, B, W be defined as usual. Let $\mathbf{i} = (i_1, \ldots, i_l)$ be a reduced word of w. The Bott-Samelson variety associated to the reduced word $\mathbf{i}$ is

$$\mathrm{P}_{i_1} \underset{\mathrm{B}}{\times} \mathrm{P}_{i_2} \underset{\mathrm{B}}{\times} \ldots \underset{\mathrm{B}}{\times} \mathrm{P}_{i_l} \Big/ \mathrm{B}$$

  where $\mathrm{P}_i = \mathrm{B} \sqcup \mathrm{B}s_i\mathrm{B}$.

- Note that

$$\mathrm{P}_{i_1} \underset{\mathrm{B}}{\times} \ldots \underset{\mathrm{B}}{\times} \mathrm{P}_{i_l} = \bigsqcup_{\mathbf{j} \subset \mathbf{i}} (\mathrm{B}s_{j_1}\mathrm{B}) \underset{\mathrm{B}}{\times} \ldots \underset{\mathrm{B}}{\times} (\mathrm{B}s_{j_m}\mathrm{B})$$

  where $\mathbf{j} = (j_1, \ldots, j_m)$ runs over all subwords of $\mathbf{i}$ (not necessarily reduced). These can be thought of as "Bott-Samelson cell".

- Alternatively one can think of an element of $(\mathrm{B}s_{j_1}\mathrm{B}) \underset{\mathrm{B}}{\times} \ldots \underset{\mathrm{B}}{\times} (\mathrm{B}s_{j_m}\mathrm{B})$ as a sequence of flags that satisfies the relative position conditions imposed by the simple reflections $s_{j_1}, s_{j_2}, \ldots, s_{j_m}$. So a "double Bott-Samelson cell" will then be two sequences of flags that satisfy two sequences of relative position conditions imposed by two words $\mathbf{i}$ and $\mathbf{j}$.

Definition

- Let G be the Kac-Peterson group (the smallest Kac-Moody group) associated to a symmetrizable generalized Cartan matrix and let $B_{\pm}$ be the two opposite Borel subgroups.

Definition

- Let G be the Kac-Peterson group (the smallest Kac-Moody group) associated to a symmetrizable generalized Cartan matrix and let $B_\pm$ be the two opposite Borel subgroups.
- Let $\mathcal{B}_\pm = \{$Borel subgroups that are conjugates of $B_\pm\}$. Bruhat decomposition implies that the G-orbits in $\mathcal{B}_+ \times \mathcal{B}_+$ and $\mathcal{B}_- \times \mathcal{B}_-$ are parametrized by the Weyl group W.

Definition

- Let G be the Kac-Peterson group (the smallest Kac-Moody group) associated to a symmetrizable generalized Cartan matrix and let $B_\pm$ be the two opposite Borel subgroups.
- Let $\mathcal{B}_\pm = \{$Borel subgroups that are conjugates of $B_\pm\}$. Bruhat decomposition implies that the G-orbits in $\mathcal{B}_+ \times \mathcal{B}_+$ and $\mathcal{B}_- \times \mathcal{B}_-$ are parametrized by the Weyl group W.

Notation

- We use superscript to denote Borel subgroups in $\mathcal{B}_+$, e.g. B^0, B^1, etc.
- We use subscript to denote Borel subgroups in $\mathcal{B}_-$, e.g. B_0, B_1, etc.
- We write $B^0 \xrightarrow{w} B^1$ if (B^0, B^1) is in the w-orbit in $\mathcal{B}_+ \times \mathcal{B}_+$.
- We write $B_0 \xrightarrow{w} B_1$ if (B_0, B_1) is in the w-orbit in $\mathcal{B}_- \times \mathcal{B}_-$.
- We write $B_0 \text{ ——— } B^0$ if $(B_0, B^0) = (gB_-, gB_+)$ for some $g \in G$.

Definition

- Let G be the Kac-Peterson group (the smallest Kac-Moody group) associated to a symmetrizable generalized Cartan matrix and let $\mathrm{B}_\pm$ be the two opposite Borel subgroups.
- Let $\mathcal{B}_\pm = \{$Borel subgroups that are conjugates of $\mathrm{B}_\pm\}$. Bruhat decomposition implies that the G-orbits in $\mathcal{B}_+ \times \mathcal{B}_+$ and $\mathcal{B}_- \times \mathcal{B}_-$ are parametrized by the Weyl group W.

Notation

- We use superscript to denote Borel subgroups in $\mathcal{B}_+$, e.g. B^0, B^1, etc.
- We use subscript to denote Borel subgroups in $\mathcal{B}_-$, e.g. B_0, B_1, etc.
- We write $\mathrm{B}^0 \xrightarrow{w} \mathrm{B}^1$ if $\left(\mathrm{B}^0, \mathrm{B}^1\right)$ is in the w-orbit in $\mathcal{B}_+ \times \mathcal{B}_+$.
- We write $\mathrm{B}_0 \xrightarrow{w} \mathrm{B}_1$ if $(\mathrm{B}_0, \mathrm{B}_1)$ is in the w-orbit in $\mathcal{B}_- \times \mathcal{B}_-$.
- We write $\mathrm{B}_0 \frac{\quad\quad}{} \mathrm{B}^0$ if $\left(\mathrm{B}_0, \mathrm{B}^0\right) = (g\mathrm{B}_-, g\mathrm{B}_+)$ for some $g \in \mathrm{G}$.

- $\mathrm{B}_+ s_i \mathrm{B}_+ / \mathrm{B}_+$ can be thought of as the moduli space of B_1 satisfying $\mathrm{B}_0 \xrightarrow{s_i} \mathrm{B}_1$ for a fixed B_0.

Definition

Definition

Let b and d be two positive braids in the associated braid group. First choose a word $(i_1, i_2, \ldots, i_m)$ for b and a word $(j_1, j_2, \ldots, j_n)$ for d. The *undecorated double Bott-Samelson cell* $\mathrm{Conf}^b_d(\mathcal{B})$ is defined to be

$$\left\{ \begin{array}{ccccccc} \mathsf{B}^0 & \xrightarrow{s_{i_1}} & \mathsf{B}^1 & \xrightarrow{s_{i_2}} & \cdots & \xrightarrow{s_{i_m}} & \mathsf{B}^m \\ \Big| & & & & & & \Big| \\ \mathsf{B}_0 & \xrightarrow[s_{j_1}]{} & \mathsf{B}_1 & \xrightarrow[s_{j_2}]{} & \cdots & \xrightarrow[s_{j_n}]{} & \mathsf{B}_n \end{array} \right\} \Big/ G$$

Definition

Definition

Let b and d be two positive braids in the associated braid group. First choose a word $(i_1, i_2, \ldots, i_m)$ for b and a word $(j_1, j_2, \ldots, j_n)$ for d. The *undecorated double Bott-Samelson cell* $\mathrm{Conf}_d^b(\mathcal{B})$ is defined to be

$$\left\{ \begin{array}{ccccccc} \mathrm{B}^0 & \xrightarrow{s_{i_1}} & \mathrm{B}^1 & \xrightarrow{s_{i_2}} & \ldots & \xrightarrow{s_{i_m}} & \mathrm{B}^m \\ | & & & & & & | \\ \mathrm{B}_0 & \xrightarrow[s_{j_1}]{} & \mathrm{B}_1 & \xrightarrow[s_{j_2}]{} & \ldots & \xrightarrow[s_{j_n}]{} & \mathrm{B}_n \end{array} \right\} \Big/ G$$

Remark

The resulting space does not depend on the choice of words for b and d.

Definition

- Let $\mathrm{U}_\pm := [\mathrm{B}_\pm, \mathrm{B}_\pm]$ and define *decorated flag varieties* $\mathcal{A}_\pm := \mathrm{G}/\mathrm{U}_\pm$. We denote decorated flags with a symbol A instead of B.

Definition

The *decorated double Bott-Samelson cell* $\mathrm{Conf}^b_d(\mathcal{A})$ is defined to be

$$\left\{ \begin{array}{ccccccc} \mathrm{A}^0 & \xrightarrow{s_{i_1}} & \mathrm{B}^1 & \xrightarrow{s_{i_2}} & \dots & \xrightarrow{s_{i_m}} & \mathrm{B}^m \\ | & & & & & & | \\ \mathrm{B}_0 & \xrightarrow[s_{j_1}]{} & \mathrm{B}_1 & \xrightarrow[s_{j_2}]{} & \dots & \xrightarrow[s_{j_n}]{} & \mathrm{A}_n \end{array} \right\} \Big/ G$$

Definition

- Let $U_\pm := [B_\pm, B_\pm]$ and define *decorated flag varieties* $\mathcal{A}_\pm := G/U_\pm$. We denote decorated flags with a symbol A instead of B.

Definition

The *decorated double Bott-Samelson cell* $\mathrm{Conf}^b_d(\mathcal{A})$ is defined to be

$$\left\{ \begin{array}{ccccccc} A^0 & \xrightarrow{s_{i_1}} & B^1 & \xrightarrow{s_{i_2}} & \dots & \xrightarrow{s_{i_m}} & B^m \\ \Big| & & & & & & \Big| \\ B_0 & \xrightarrow[s_{j_1}]{} & B_1 & \xrightarrow[s_{j_2}]{} & \dots & \xrightarrow[s_{j_n}]{} & A_n \end{array} \right\} \Big/ G$$

- Decorated double Bott-Samelson cell can be viewed as a generalization of double Bruhat cells $B_+uB_+ \cap B_-vB_-$. Double Bruhat cells are examples of cluster varieties and are studied by Berenstein, Fomin, and Zelevinsky [BFZ05], Fock and Goncharov [FG06], and many others.

Definition

- Let $\mathrm{U}_\pm := [\mathrm{B}_\pm, \mathrm{B}_\pm]$ and define *decorated flag varieties* $\mathcal{A}_\pm := \mathrm{G}/\mathrm{U}_\pm$. We denote decorated flags with a symbol A instead of B.

Definition

The *decorated double Bott-Samelson cell* $\mathrm{Conf}^b_d(\mathcal{A})$ is defined to be

$$\left\{ \begin{array}{ccccccc} \mathrm{A}^0 & \xrightarrow{s_{i_1}} & \mathrm{B}^1 & \xrightarrow{s_{i_2}} & \dots & \xrightarrow{s_{i_m}} & \mathrm{B}^m \\ | & & & & & & | \\ \mathrm{B}_0 & \xrightarrow[s_{j_1}]{} & \mathrm{B}_1 & \xrightarrow[s_{j_2}]{} & \dots & \xrightarrow[s_{j_n}]{} & \mathrm{A}_n \end{array} \right\} \Big/ G$$

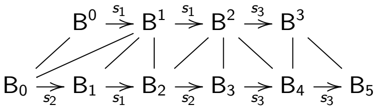
- Decorated double Bott-Samelson cell can be viewed as a generalization of double Bruhat cells $\mathrm{B}_+ u \mathrm{B}_+ \cap \mathrm{B}_- v \mathrm{B}_-$. Double Bruhat cells are examples of cluster varieties and are studied by Berenstein, Fomin, and Zelevinsky [BFZ05], Fock and Goncharov [FG06], and many others.

Theorem (Shen-W.)

The decorated double Bott-Samelson cells $\mathrm{Conf}^b_d(\mathcal{A})$ are smooth affine varieties.

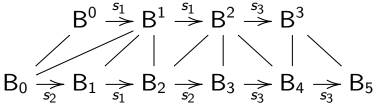
Cluster Structures

- We equip each double Bott-Samelson cell (both undecorated and decorated) with an atlas of algebraic torus charts, parametrized by a choice of words for b and d and a triangulation of the "trapezoid".



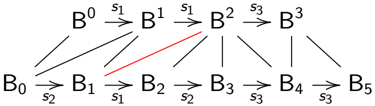
Cluster Structures

- We equip each double Bott-Samelson cell (both undecorated and decorated) with an atlas of algebraic torus charts, parametrized by a choice of words for b and d and a triangulation of the "trapezoid".

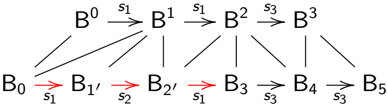


- There are two kinds of moves available to us:

diagonal flipping



Braid move



Cluster Structures

- We actually consider two versions of decorated double Bott-Samelson cells, one for G_{sc} and one for G_{ad} (analogues of the simply-connected form and the adjoint form in the semisimple cases).

Cluster Structures

- We actually consider two versions of decorated double Bott-Samelson cells, one for G_{sc} and one for G_{ad} (analogues of the simply-connected form and the adjoint form in the semisimple cases).
- The natural projection $\mathsf{G}_{\mathrm{sc}} \to \mathsf{G}_{\mathrm{ad}}$ gives rise to natural projection maps $\mathcal{A}_{\mathrm{sc}} \to \mathcal{A}_{\mathrm{ad}}$ and $p : \mathrm{Conf}^b_d(\mathcal{A}_{\mathrm{sc}}) \to \mathrm{Conf}^b_d(\mathcal{A}_{\mathrm{ad}})$.

Cluster Structures

- We actually consider two versions of decorated double Bott-Samelson cells, one for G_{sc} and one for G_{ad} (analogues of the simply-connected form and the adjoint form in the semisimple cases).
- The natural projection $\mathsf{G}_{\mathrm{sc}} \to \mathsf{G}_{\mathrm{ad}}$ gives rise to natural projection maps $\mathcal{A}_{\mathrm{sc}} \to \mathcal{A}_{\mathrm{ad}}$ and $p : \mathrm{Conf}^b_d(\mathcal{A}_{\mathrm{sc}}) \to \mathrm{Conf}^b_d(\mathcal{A}_{\mathrm{ad}})$.

Theorem (Shen-W.)

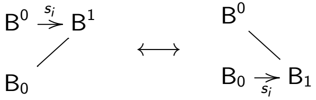
The atlas of algebraic torus charts are related by birational maps called cluster mutations. These charts equips $\mathcal{O}\left(\mathrm{Conf}^b_d(\mathcal{A}_{\mathrm{sc}})\right)$ with the structure of an upper cluster algebra, and equips $\mathcal{O}\left(\mathrm{Conf}^b_d(\mathcal{A}_{\mathrm{ad}})\right)$ with the structure of an upper cluster Poisson algebra. The pair $\left(\mathrm{Conf}^b_d(\mathcal{A}_{\mathrm{sc}}), \mathrm{Conf}^b_d(\mathcal{A}_{\mathrm{ad}})\right)$ form a Fock-Goncharov cluster ensemble.

Reflection Maps between double Bott-Samelson cells

- We constructed biregular maps called *reflection maps*:

$$\mathrm{Conf}_d^{bs_i}(\mathcal{B}) \longleftrightarrow \mathrm{Conf}_{ds_i}^{b}(\mathcal{B}) \qquad \mathrm{Conf}_d^{s_i b}(\mathcal{B}) \longleftrightarrow \mathrm{Conf}_{s_i d}^{b}(\mathcal{B}).$$

They are induced by moves that look like the following:

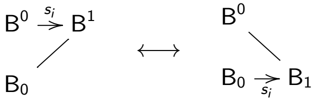


Reflection Maps between double Bott-Samelson cells

- We constructed biregular maps called *reflection maps*:

$$\mathrm{Conf}_d^{bs_i}(\mathcal{B}) \longleftrightarrow \mathrm{Conf}_{ds_i}^{b}(\mathcal{B}) \qquad \mathrm{Conf}_d^{s_i b}(\mathcal{B}) \longleftrightarrow \mathrm{Conf}_{s_i d}^{b}(\mathcal{B}).$$

  They are induced by moves that look like the following:



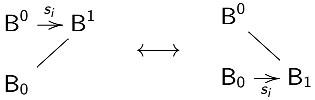
- These reflection maps are Poisson and respect the cluster structures.

Reflection Maps between double Bott-Samelson cells

- We constructed biregular maps called *reflection maps*:

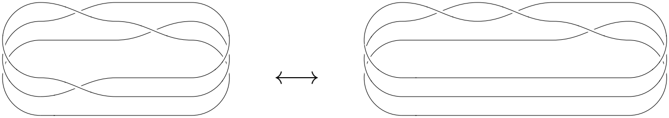
$$\mathrm{Conf}^{bs_i}_{d}(\mathcal{B}) \longleftrightarrow \mathrm{Conf}^{b}_{ds_i}(\mathcal{B}) \qquad \mathrm{Conf}^{s_i b}_{d}(\mathcal{B}) \longleftrightarrow \mathrm{Conf}^{b}_{s_i d}(\mathcal{B}).$$

  They are induced by moves that look like the following:



- These reflection maps are Poisson and respect the cluster structures.
- One can think of such reflection maps as movement of tangles in a link.

$$\mathrm{Conf}^{s_1 s_2}_{s_1}(\mathcal{B}) \quad \longleftrightarrow \quad \mathrm{Conf}^{s_1 s_1 s_2}_{e}(\mathcal{B})$$



Cluster Donaldson-Thomas Transformation

- One important conjecture in cluster theory is the Fock-Goncharov cluster duality [FG09], which conjectures the existence of canonical bases in an upper cluster algebra and its corresponding upper cluster Poisson algebra.

Cluster Donaldson-Thomas Transformation

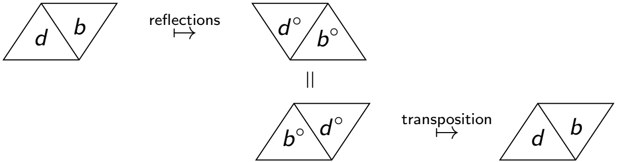
- One important conjecture in cluster theory is the Fock-Goncharov cluster duality [FG09], which conjectures the existence of canonical bases in an upper cluster algebra and its corresponding upper cluster Poisson algebra.
- Part of a sufficient condition [GHKK18] [GS18] of the duality conjecture is the existence of the cluster Donaldson-Thomas transformation.

Cluster Donaldson-Thomas Transformation

- One important conjecture in cluster theory is the Fock-Goncharov cluster duality [FG09], which conjectures the existence of canonical bases in an upper cluster algebra and its corresponding upper cluster Poisson algebra.
- Part of a sufficient condition [GHKK18] [GS18] of the duality conjecture is the existence of the cluster Donaldson-Thomas transformation.

Theorem (Shen-W.)

Cluster Donaldson-Thomas transformations exist on double Bott-Samelson cells and are given by compositions of reflection maps and a transposition map. Reflection maps intertwine the cluster Donaldson-Thomas transformations on different double Bott-Samelson cells. By verifying the sufficient condition, we prove the cluster duality conjecture for double Bott-Samelson cells.



Periodicity of DT in the Semisimple Case

- For the rest of the talk, let G be semisimple and let w_0 denote the longest Weyl group element.

Periodicity of DT in the Semisimple Case

- For the rest of the talk, let G be semisimple and let w_0 denote the longest Weyl group element.

Theorem (Shen-W.)

Let G be a semisimple group. Let b be a positive braid and let m, n be two positive integers such that $b^m = w_0^{2n}$. Then the order of the cluster Donaldson-Thomas transformation of $\mathrm{Conf}^e_b(\mathcal{B})$ is finite and divides $2(m+n)$.

Periodicity of DT in the Semisimple Case

- For the rest of the talk, let G be semisimple and let w_0 denote the longest Weyl group element.

Theorem (Shen-W.)

Let G be a semisimple group. Let b be a positive braid and let m, n be two positive integers such that $b^m = w_0^{2n}$. Then the order of the cluster Donaldson-Thomas transformation of $\mathrm{Conf}^e_b(\mathcal{B})$ is finite and divides $2(m+n)$.

Example

Suppose $G = \mathrm{SL}_3$ and $b = s_1 s_2 s_1 s_2$. Then $b^3 = w_0^4$ in the braid group, and therefore $\mathrm{DT}^{10} = \mathrm{Id}$ on $\mathrm{Conf}^e_b(\mathcal{B})$. Intertwining by a reflection map, this computation also implies that $\mathrm{DT}^{10} = \mathrm{Id}$ on $\mathrm{Conf}^{s_1}_{w_0}(\mathcal{B})$ in the double Bruhat cell case as well.

New Proof of Zamolodchikov's Periodicity Conjecture

- One version of the conjecture (formulated by Keller [Kel13]) is about the periodicities of the Donaldson-Thomas transformations associated to products of two Dynkin diagrams.

Theorem (Keller)

Let D and D' be Dynkin quivers with Coxeter numbers h and h'. Then $\mathrm{DT}_{D\boxtimes D'}^{2(h+h')} = \mathrm{Id}$.

New Proof of Zamolodchikov's Periodicity Conjecture

- One version of the conjecture (formulated by Keller [Kel13]) is about the periodicities of the Donaldson-Thomas transformations associated to products of two Dynkin diagrams.

Theorem (Keller)

Let D and D' be Dynkin quivers with Coxeter numbers h and h'. Then $\mathrm{DT}_{D\boxtimes D'}^{2(h+h')} = \mathrm{Id}$.

- Using our result on the periodicity of DT on double Bott-Samelson cells, we can give a new geometric proof of the periodicity conjecture in the case of $D \boxtimes A_n$.

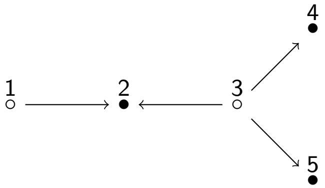
New Proof of Zamolodchikov's Periodicity Conjecture

- One version of the conjecture (formulated by Keller [Kel13]) is about the periodicities of the Donaldson-Thomas transformations associated to products of two Dynkin diagrams.

Theorem (Keller)

Let D and D' be Dynkin quivers with Coxeter numbers h and h'. Then $\mathrm{DT}_{D\boxtimes D'}^{2(h+h')} = \mathrm{Id}$.

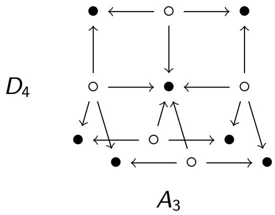
- Using our result on the periodicity of DT on double Bott-Samelson cells, we can give a new geometric proof of the periodicity conjecture in the case of $D \boxtimes A_n$.
- Give D a bipartite coloring.



$$b = s_2 s_4 s_5, \quad w = s_1 s_3$$

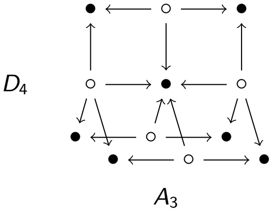
New Proof of Zamolodchikov's Periodicity Conjecture

- Consider the double Bott-Samelson cell $\mathrm{Conf}^{bwb\cdots}_{wbw\cdots}(\mathcal{B})$, where the number of b and w in each braid sum up to $n+1$.



New Proof of Zamolodchikov's Periodicity Conjecture

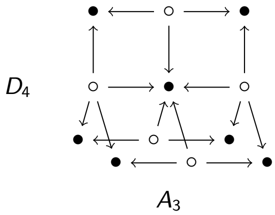
- Consider the double Bott-Samelson cell $\mathrm{Conf}^{bwb\ldots}_{wbw\ldots}(\mathcal{B})$, where the number of b and w in each braid sum up to $n+1$.



- Note that $bw = c$ and $\mathrm{Conf}^{bwb\ldots}_{wbw\ldots}(\mathcal{B}) \cong \mathrm{Conf}^{e}_{c^{n+1}}(\mathcal{B})$.

New Proof of Zamolodchikov's Periodicity Conjecture

- Consider the double Bott-Samelson cell $\mathrm{Conf}^{bwb\dots}_{wbw\dots}(\mathcal{B})$, where the number of b and w in each braid sum up to $n+1$.



- Note that $bw = c$ and $\mathrm{Conf}^{bwb\dots}_{wbw\dots}(\mathcal{B}) \cong \mathrm{Conf}^{e}_{c^{n+1}}(\mathcal{B})$.
- Let h be the Coxeter number of D. Since $\left(c^{n+1}\right)^h = w_0^{2(n+1)}$, our result implies that $\mathrm{DT}^{2(h+n+1)}_{D\boxtimes A_n} = \mathrm{Id}$.

Thank you!

Bibliography

A. Berenstein, S. Fomin, and A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52, arXiv:math/0305434.

V. Fock and A. Goncharov, *Cluster X-varieties, amalgamation and Poisson-Lie groups*, Algebraic Geometry and Number Theory, In honor of Vladimir Drinfeld's 50th birthday, Birkhäuser Boston (2006), 27–68, arXiv:math/0508408.

______, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci. Éc. Norm. Supér. **42** (2009), no. 6, 865–930, arXiv:math/0311245.

M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31** (2018), no. 2, 497–608, arXiv:1411.1394.

A. Goncharov and L. Shen, *Donaldson-Thomas transformations for moduli spaces of G-local systems*, Adv. Math. **327** (2018), 225–348, arXiv:1602.06479.

B. Keller, *The periodicty conjecture for pairs of Dynkin diagrams*, Ann. Of Math. **177** (2013), no. 1, 111–170, arXiv:1001.1531.