Deformations of Gorenstein-projective Modules over Nakayama and Triangular Matrix Algebras

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- In this talk, we assume that $\Bbbk$ is an algebraically closed field of arbitrary characteristic and that all our modules are finite-dimensional over $\Bbbk$.

Definition 1. (E. ENOCHS, O. JENDA, 1995) Let Λ be a finite dimensional $\Bbbk$-algebra. A Λ-module V is said to be **Gorenstein-projective** if there exists an acyclic complex of projective Λ-modules

$$P^\bullet : \cdots \to P^{-2} \xrightarrow{\delta^{-2}} P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\delta^0} P^1 \xrightarrow{\delta^1} P^2 \to \cdots$$

such that $\mathrm{Hom}_\Lambda(P^\bullet, \Lambda)$ is also acyclic and $V = \mathrm{coker}\,\delta^0$.

- We denote by Λ-Gproj the category of Gorenstein-projective left Λ-modules, and by Λ-$\underline{\text{Gproj}}$ its stable category.
- Λ is self-injective if and only if every left Λ-module is Gorenstein-projective.
- If Λ has finite global dimension, then every Gorenstein projective left Λ-module is projective.
- There are finite dimensional $\Bbbk$-algebras Λ of infinite global dimension such that every Gorenstein-projective left Λ-module is projective (see e.g. (X.-W. CHEN & Y. YE, 2014)).
- (R. -O. BUCHWEITZ, 1987) If Λ is Gorenstein (i.e. Λ has finite injective dimension as a left and right Λ-module), then

$$\mathcal{D}_{\text{sg}}(\Lambda\text{-mod}) = \mathcal{D}^b(\Lambda\text{-mod})/\mathcal{K}^b(\Lambda\text{-proj}) = \Lambda\text{-}\underline{\text{Gproj}}.$$

Gorenstein-projective Modules

Finitely generated Gorenstein-projective modules are also known as:

- **Modules of Gorenstein-dimension zero** (M. Auslander & M. Bridger, 1969).
- **Maximal Cohen-Macaulay modules** provided that Λ is Gorenstein (R.-O. Buchweitz, 1987).
- **Cohen-Macaulay modules** (A. Beligiannis & I. Reiten, 2001).
- **Totally reflexive modules** (L. Avramov & A. Martsinkovsky, 2002).

Explicit descriptions of finitely generated Gorenstein-projective modules have been found for the following classes of finite dimensional $\Bbbk$-algebras (this list may be incomplete).

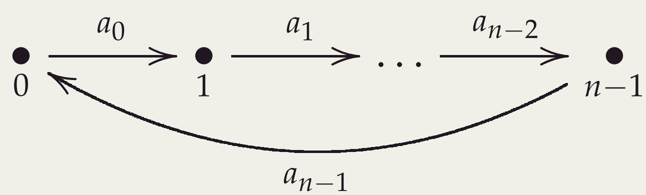
- Basic connected Nakayama algebras with no simple projective modules (C.M. Ringel, 2013).
- Triangular matrix algebras (P. Zhang, 2013).
- Gentle algebras (M. Kalck, 2015).
- 2-Calabi-Yau tilted algebras (A. García Elsener, R. Schiffler, 2017).
- Monomial algebras (X.W. Chen, D. Shen, G. Zhou, 2018)

GORENSTEIN-PROJECTIVE MODULES OVER NAKAYAMA ALGEBRAS

- From now on we assume that Λ is a basic connected finite-dimensional $\Bbbk$-algebra, i.e., Λ is of the form $\Bbbk Q/I$, where Q is a finite quiver and I is an admissible ideal of $\Bbbk Q$.
- Recall that Λ is said to be a **Nakayama algebra** if every left or right indecomposable projective Λ-module has a unique composition series.

Theorem 2. *Λ is a Nakayama algebra with no simple projective modules if and only if $\Lambda = \Bbbk Q/I$, where Q is the quiver:*

C_n :



for some $n \geq 1$.

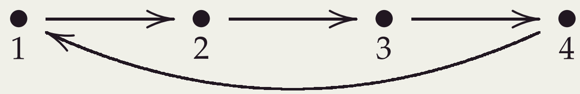
Theorem 3. (C. M. RINGEL, 2013) *Let Λ be a Nakayama algebra with no simple projective modules. A left Λ-module V is Gorenstein-projective if and only if there exists an exact sequence of Λ-modules*

$$0 \to V \to P_{n-1} \to \cdots \to P_0 \to V \to 0,$$

where each P_i is a **minimal projective** *Λ-module, i.e., no proper non-zero submodule of P_i is projective.*

Running Example: The Nakayama algebra Λ with admissible sequence $\mathbf{c}(\Lambda) = (10, 10, 9, 9)$

Consider the Nakayama algebra whose quiver and admissible sequence are given by



$$\mathbf{c}(\Lambda) = (10, 10, 9, 9).$$

- Note that Λ is a non-self-injective $\mathbb{k}$-algebra.
- The minimal projective left Λ-modules are given by $P_1 = S_1^{[10]}$, $P_3 = S_3^{[9]}$ and $P_4 = S_4^{[9]}$.
- For example $S_1^{[2]}$ is Gorenstein-projective for we have a exact sequence of left Λ-modules

$$0 \to S_1^{[2]} \to P_1 \to P_4 \to P_4 \to P_3 \to P_3 \to P_1 \to S_1^{[2]} \to 0.$$

Definition 4. (C. M. Ringel. 2013) Let Λ be a basic Nakayama algebra without simple projective Λ-modules.

- We denote by $\mathscr{C}(\Lambda)$ be the subcategory of Λ-mod whose indecomposable objects are the indecomposable non-projective Gorenstein-projective left Λ-modules as well as their corresponding projective covers. We call $\mathscr{C}(\Lambda)$ the **Gorenstein core** of Λ.
- We also denote by $\mathscr{E}(\Lambda)$ be the class of non-zero indecomposable Gorenstein-projective Λ-modules E such that no proper non-zero factor module of E is a Gorenstein-projective Λ-module. Then the objects in $\mathscr{E}(\Lambda)$ are the simple objects in $\mathscr{C}(\Lambda)$ called the **elementary** Gorenstein-projective modules of Λ.

Theorem 5. (C. M. RINGEL. 2013) *Let Λ be a basic Nakayama algebra without simple projective Λ-modules and denote by $s = s(\Lambda)$ the number of isomorphism classes of simple left Λ-modules.*

(i) *Every object V in $\mathscr{C}(\Lambda))$ has a filtration with composition factors in $\mathscr{E}(\Lambda)$. Thus $\mathscr{C}(\Lambda)$ is an* **abelian length category** *in the sense of* (P. GABRIEL, 1973).

(ii) *There exists a basic connected* **self-injective** *Nakayama algebra Λ' such that the categories $\mathscr{C}(\Lambda)$ and Λ'-*mod *are equivalent.*

(iii) *If $\mathscr{E}(\Lambda) = \{E_1, \ldots, E_g\}$ and p_i is the length of the projective Λ-module cover $P(E_i)$ of E_i for all $1 \leq i \leq g$ with $s < p_i$, then Λ' has exactly $e = e(\Lambda') = g$ isomorphism classes of simple Λ'-modules and the Loewy length $\ell\ell(\Lambda')$ of Λ' is given by*

$$\ell\ell(\Lambda') = \frac{1}{s}\sum_{i=1}^{g} p_i.$$

The Auslander-Reiten Quiver of The Nakayama Algebra with admissible sequence $\mathbf{c}(\Lambda) = (10, 10, 9, 9)$

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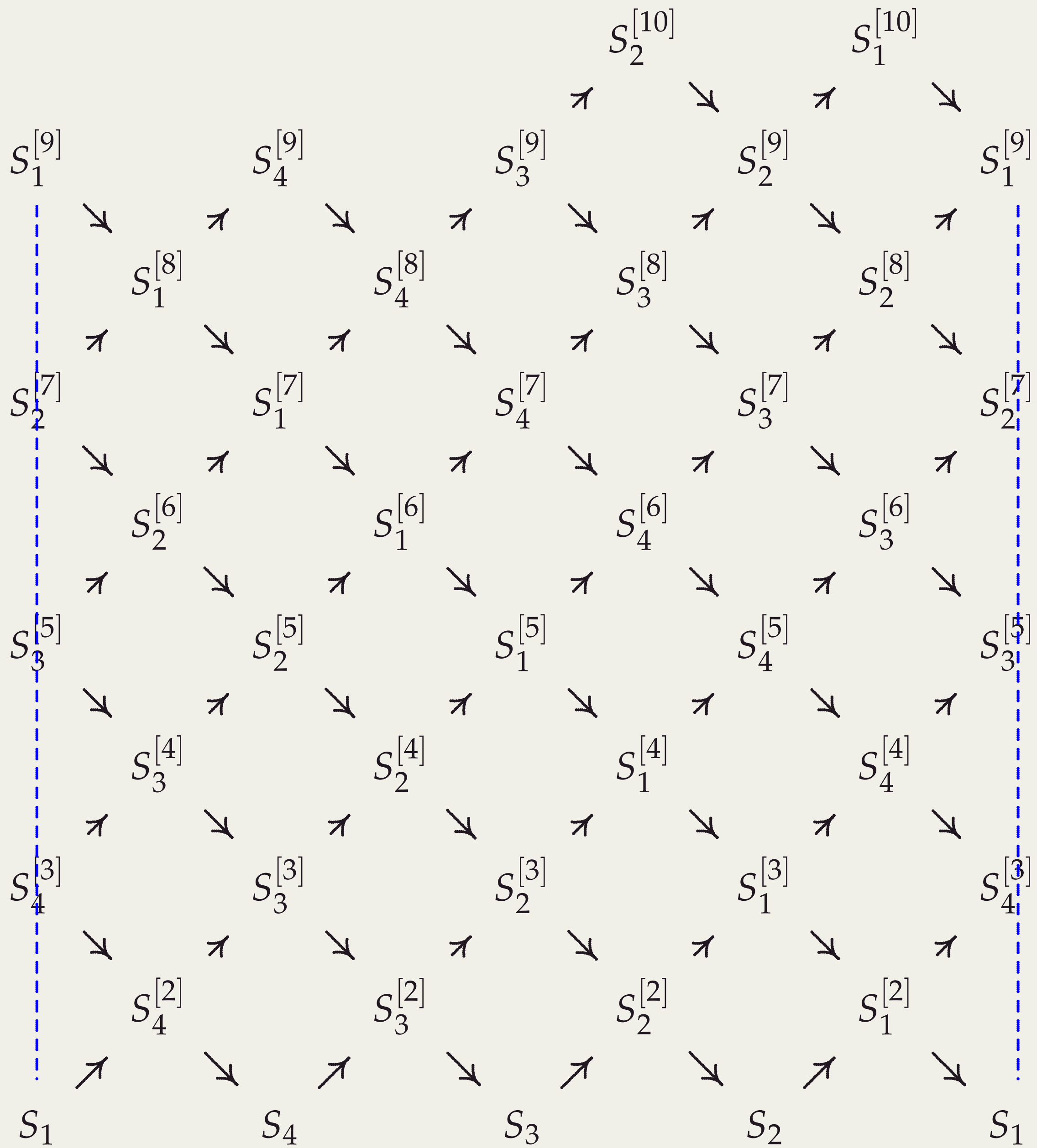


The Nakayama Algebra Λ with four vertices and admissible sequence $\mathbf{c}(\Lambda) = (10, 10, 9, 9)$

$$\mathscr{E}(\Lambda) = \{S_1^{[2]}, S_3, S_4\}$$

$$\mathscr{X}(\Lambda) = \{S_1, S_3, S_4\}$$

$$P(\mathscr{X}(\Lambda)) = S_1^{[10]} \oplus S_3^{[9]} \oplus S_4^{[9]}$$

$$\Lambda' = \mathrm{End}_\Lambda(P(\mathscr{X}(\Lambda)))^{\mathrm{op}}$$

$$\ell\ell(\Lambda') = \tfrac{1}{4}(10 + 9 + 9) = 7$$

$$\mathbf{c}(\Lambda') = (7, 7, 7)$$

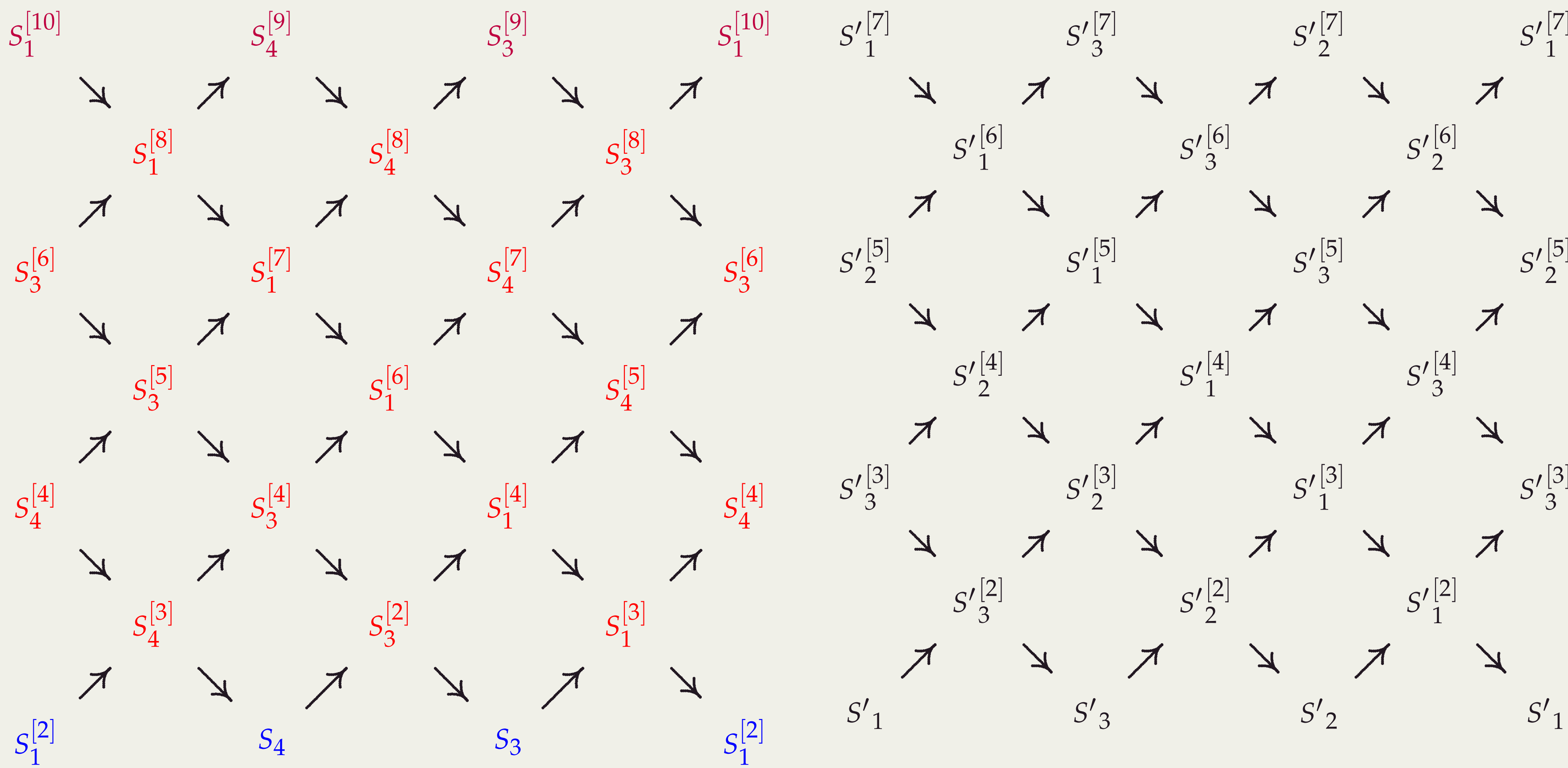


Figure 1: The Auslander-Reiten quivers of $\mathscr{C}(\Lambda)$ and Λ'-mod

Example 6. Note e.g. that $|S_1^{[8]}|_{\mathscr{C}(\Lambda)} = 6$ for the composition factors of $S_1^{[8]}$ in $\mathscr{E}(\Lambda)$ are $S_1^{[2]}$, S_3, S_4, $S_1^{[2]}$, S_3, S_4.

Theorem 7. *Let Λ be a finite dimensional $\Bbbk$-algebra and V a left Λ-module.*

(i) *V always has a* **versal deformation ring** *$R(\Lambda, V)$ which is a local complete commutative Noetherian $\Bbbk$-algebra with residue filed $\Bbbk$, and which is further universal provided that $\mathrm{End}_\Lambda(V) = \Bbbk$. Moreover, versal deformation rings are invariants under Morita equivalence* (F. M. BLEHER-JVM, 2012).

(ii) *Versal deformation rings of non-projective modules are invariant under* **stable equivalence of Morita type** *between self-injective finite dimensional $\Bbbk$-algebras* (F. M. BLEHER JVM, 2017).

(iii) *If Λ is Frobenius and V is non-projective, then the versal deformation rings $R(\Lambda, V)$ and $R(\Lambda, \Omega V)$ are isomorphic* (F. M. BLEHER, D. WACKWITZ, 2019).

(iv) *Versal deformation rings of Gorenstein-projective modules are invariant under* **singular equivalences of Morita type** *(in the sense of X. W. Chen & L. G. Sun) between Gorenstein $\Bbbk$-algebras* (V. BEKKERT, H. GIRALDO, JVM,2019).

(v) *If Λ is a* **monomial algebra in which there is no overlap** *(in the sense of (X. W. Chen, D. Shen & G. Zhou, 2018)) and V is Gorenstein-projective, then $R(\Lambda, V)$ is universal and isomorphic to $\Bbbk$ or to $\Bbbk[\![t]\!]/(t^2)$* (V. BEKKERT, H. GIRALDO, JVM,2019).

Theorem 8. (F. M. Bleher and D. J. Wackwitz, 2019) *Assume that Λ is a self-injective (so Frobenius) Nakayama $\Bbbk$-algebra, and let V be an indecomposable non-projective left Λ-module.*

(i) *The versal deformation ring $R(\Lambda, V)$ is universal.*
(ii) *The universal deformation rings $R(\Lambda, V)$ and $R(\Lambda, \Omega V)$ are isomorphic in $\hat{\mathcal{C}}$.*
(iii) *Moreover, $R(\Lambda, V)$ is isomorphic either to $\Bbbk$, or $\Bbbk[\![t]\!]/(t^2)$, or determined as a quotient ring of a power series ring over $\Bbbk$ in finitely many variables by the shortest distance d_V of the isomorphism class of V to the boundary of the* stable *Auslander-Reiten quiver of Λ.*

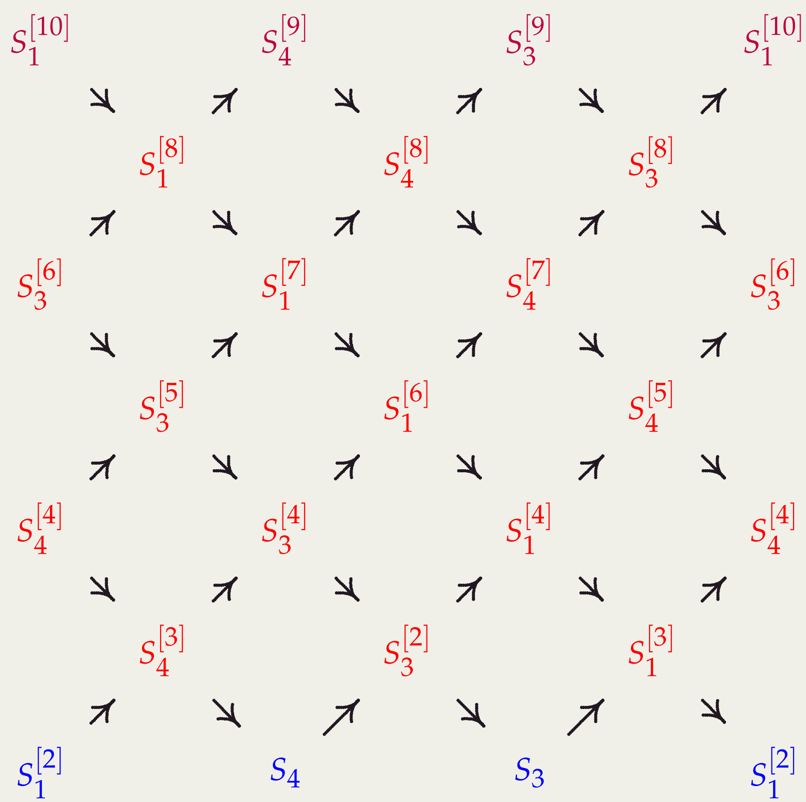
Theorem 9. (JVM, 2019) *Let Λ be a Nakayama $\Bbbk$-algebra with no simple projective modules and with $\mathscr{C}(\Lambda) \neq 0$. Let V be an indecomposable non-projective Gorentein-projective left Λ-module in $\mathscr{C}(\Lambda)$.*

(i) *The versal deformation ring $R(\Lambda, V)$ is isomorphic to $R(\Lambda', V')$ in $\hat{\mathcal{C}}$, where Λ' is a self-injective Nakayama $\Bbbk$-algebra. In particular, $R(\Lambda, V)$ is also universal and isomorphic to $R(\Lambda, \Omega V)$ in $\hat{\mathcal{C}}$.*
(ii) *Moreover, $R(\Lambda, V)$ is isomorphic either to $\Bbbk$, or $\Bbbk[\![t]\!]/(t^2)$, or determined as a quotient ring of a power series ring over $\Bbbk$ in finitely many variables by the shortest distance $d_{\mathscr{C}(\Lambda),V}$ of the isomorphism class of V to the boundary of the* stable *Auslander-Reiten quiver of $\mathscr{C}(\Lambda)$.*

Example 10. Let Λ' be the self-injective Nakayama $\mathbb{k}$-algebra with admissible sequence $\mathbf{c}(\Lambda') = (7,7,7)$ discussed before. Then the universal deformation rings $R(\Lambda', V')$ of the indecomposable Λ'-modules V' are described below, where $d_{V'}$ denotes the shortest distance of V' to the boundary of the stable Auslander-Reiten quiver of Λ'.

$d(V')$	$R(\Lambda', V')$
–	$\mathbb{k}$
0	$\mathbb{k}$
1	$\mathbb{k}$
2	$\mathbb{k}[\![t]\!]/(t^2)$
2	$\mathbb{k}[\![t]\!]/(t^2)$
1	$\mathbb{k}$
0	$\mathbb{k}$

Let Λ be cycle Nakayama $\Bbbk$-algebra with admissible sequence $\mathbf{c}(\Lambda) = (10, 10, 9, 9)$ discussed before. Then the universal deformation rings $R(\Lambda, V)$ of the indecomposable Gorenstein-projective left Λ-modules V in $\mathscr{C}(\Lambda)$ are described below, where $d_{\mathscr{C}(\Lambda),V}$ denotes the shortest distance of V to the boundary of the stable Auslander-Reiten quiver of $\mathscr{C}(\Lambda)$.



$d_{\mathscr{C}(\Lambda),V}$	$R(\Lambda, V)$
–	$\Bbbk$
0	$\Bbbk$
1	$\Bbbk$
2	$\Bbbk[\![t]\!]/(t^2)$
2	$\Bbbk[\![t]\!]/(t^2)$
1	$\Bbbk$
0	$\Bbbk$

Recall that a finite dimensional $\Bbbk$-algebra Σ is a triangular matrix $\Bbbk$-algebra if Σ is of the form

$$\Sigma = \begin{pmatrix} \Lambda & B \\ 0 & \Gamma \end{pmatrix},$$

where Λ and Γ are finite dimensional $\Bbbk$-algebras and B is a Λ-Γ-bimodule.

- A left Σ-module is of the form $\begin{pmatrix} V \\ W \end{pmatrix}_f$, where V is a left Λ-module, W is a left Γ-module and $f : B \otimes_\Gamma W \to V$ is a Λ-module homomorphism.
- A Σ-module homomorphism between two left Σ-modules $\begin{pmatrix} V \\ W \end{pmatrix}_f$ and $\begin{pmatrix} V' \\ W' \end{pmatrix}_{f'}$ is of the form $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \begin{pmatrix} V \\ W \end{pmatrix}_f \to \begin{pmatrix} V' \\ W' \end{pmatrix}_{f'}$, where $\alpha : V \to V'$ is a Λ-module homomorphism, $\beta : W \to W'$ is a Γ-module homomorphism, and $f' \circ (\mathrm{id}_B \otimes \beta) = \alpha \circ f$.

Definition 11. (P. ZHANG, 2013) We say that the Λ-Γ-bimodule B is **compatible** if satisfies the following conditions:

(i) If $Q^\bullet$ is a exact sequence of projective Γ-modules, then $B \otimes_\Gamma Q^\bullet$ is also exact.
(ii) If $P^\bullet$ is a complete Λ-projective resolution, then $\mathrm{Hom}_\Lambda(P^\bullet, B)$ is also exact.

In particular, if B has finite projective dimension as a left Λ-module and as a right Γ-module, then B is compatible.

Theorem 12. (P. ZHANG, 2013) *Let Σ be a triangular matrix $\Bbbk$ algebra as before with B a compatible Λ-Γ-bimodule, and let $\begin{pmatrix} V \\ W \end{pmatrix}_f$ be a left Σ-module.*

(i) $\begin{pmatrix} V \\ W \end{pmatrix}_f$ *is Gorenstein-projective if and only if the Λ-module homomorphism $f : B \otimes_\Gamma W \to V$ is injective, $\operatorname{coker} f$ is a Gorenstein-projective left Λ-module, and W is a Gorenstein-projective left Γ-module.*

(ii) *Moreover, V is a Gorenstein-projective left Λ-module if and only if $B \otimes_\Gamma W$ is also a Gorenstein-projective left Λ-module.*

(iii) *Assume that B is projective as a left Λ-module and Σ is Gorenstein with Γ of finite Gorenstein dimension. Then the operator*

$$i^! : \Sigma\text{-Gproj} \to \Lambda\text{-Gproj} \tag{1}$$

defined by $\begin{pmatrix} V \\ W \end{pmatrix}_f \mapsto V$. induces an equivalence of stable categories

$$i^! : \Sigma\text{-}\underline{\text{Gproj}} \to \Lambda\text{-}\underline{\text{Gproj}} \tag{2}$$

whose quasi-inverse is given by the functor $i_* : \Lambda\text{-}\underline{\text{Gproj}} \to \Sigma\text{-}\underline{\text{Gproj}}$ which sends every non-projective Gorenstein-projective left Λ-module V to $\begin{pmatrix} V \\ 0 \end{pmatrix}_0$.

Versal Deformation Rings of Gorenstein-projective Modules over Triangular Matrix Algebras

Theorem 13. (JVM, 2019) *Let Σ be a triangular matrix $\Bbbk$-algebra as before. Assume that B is projective as a left Λ-module and Σ is Gorenstein with Γ of finite global dimension. Let $\begin{pmatrix} V \\ W \end{pmatrix}_f$ be a Gorenstein-projective left Σ-module. Then V is a Gorenstein-projective left Λ-module and the versal deformation rings $R\left(\Sigma, \begin{pmatrix} V \\ W \end{pmatrix}_f\right)$ and $R(\Lambda, V)$ are isomorphic in $\hat{\mathcal{C}}$.*

Example 14. (B. L. Xiong, P. Zhang, 2012) Let Σ be the $\Bbbk$-algebra defined by the quiver with relations

$$Q: \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet} \circlearrowleft \gamma \qquad\qquad \rho = \{\gamma^3\}.$$

- Then Σ is the triangular matrix $\Bbbk$-algebra as before, where $\Lambda = \Bbbk[x]/(x^3)$, Γ is given by the quiver $\underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet}$ and $B = e_3\Sigma(1-e_3)$. Moreover, $B \cong \Lambda \oplus \Lambda$ as left Λ-modules, and $B \cong e_2\Gamma \oplus e_2\Gamma$ as right Γ-modules.
- The indecomposable non-projective Gorenstein-projective left Σ-modules are given by the following representations:

$$U_1 = 0 \longrightarrow 0 \longrightarrow \Bbbk \circlearrowleft 0\ , \qquad\qquad U_2 = 0 \longrightarrow 0 \longrightarrow \Bbbk^2 \circlearrowleft \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

- Note that $\Omega_\Sigma U_1 = U_2$, and that $\mathrm{End}_\Sigma(U_1) = \Bbbk = \underline{\mathrm{End}}_\Sigma(U_2)$, and for $i = 1, 2$, we have that $\mathrm{Ext}^1_\Sigma(U_i, U_i) = \Bbbk$. Then the versal deformation ring $R(\Sigma, U_i)$ is universal and a quotient of $\Bbbk[\![t]\!]$.
- Let S be the unique simple left Λ-module. Since Λ is a self-injective Nakayama $\Bbbk$-algebra with Loewy length 3, we have that

$$R(\Sigma, U_i) \cong R(\Lambda, S) \cong R(\Lambda, \Omega_\Lambda S) \cong \Bbbk[\![t]\!]/(t^3).$$

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Thanks for your attention!