Generalized Igusa-Todorov Functions

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Maurice Auslander Distinguished Lectures
and International Conference

April 24-29, 2019

Preliminaries

- K. Igusa, G. Todorov. On the finitistic global dimension conjecture for Artin algebras. *Representations of algebras and related topics*, 2005.

Notation:
Λ : Artin algebra.
$mod\,\Lambda$: Category of f.g. Λ-modules.
$\mathcal{P}$: Full subcategory of projective Λ-modules.
$fin.dim(\Lambda) :=$ sup$\{pd\,M \ : \ M \in mod\,\Lambda$ and $pd\,M < \infty\}$.
$fin.dim(\mathcal{D}) :=$ sup$\{pd\,Z \ : \ Z \in \mathcal{D}$ and $pd\,Z < \infty\}$, for any class $\mathcal{D} \subset mod\,\Lambda$.

Preliminaries

$K_{\mathcal{P}}$: abelian group generated by $[X]$, where $X \in mod\,\Lambda$, modulo:

(a) $[M] = [X] + [Y]$ if $M \cong X \oplus Y$.

(b) $[P] = 0$ if $P \in \mathcal{P}$.

Define $L : K_{\mathcal{P}} \to K_{\mathcal{P}}$ as follows $L([X]) := [\Omega\, X]$.
Restrict $L : \langle X \rangle \to L(\langle X \rangle)$
Applying Fitting's lemma there is an integer $n_L(X) \geq 0$, minimal with respect to the property:

$$L : L^m(\langle X \rangle) \stackrel{\sim}{\to} L^{m+1}(\langle X \rangle), \text{ for all } m \geq n_L(X).$$

Then $\phi : mod\,\Lambda \to \mathbb{N}$ is defined by $X \mapsto n_L(X)$.

Fitting's lemma, revisited

Let $\mathcal{D}$ be a subcategory of $mod\,\Lambda$ satisfying:

- $add(\mathcal{D}) = \mathcal{D}$.
- $\Omega(\mathcal{D}) \subseteq \mathcal{D}$.

Define $\overline{L} : K_{\mathcal{D}} \to K_{\mathcal{D}}$, $x \mapsto L(x) + \langle \mathcal{D} \rangle$, where $K_{\mathcal{D}} := \dfrac{K_{\mathcal{P}}}{\langle \mathcal{D} \rangle}$.

Restrict $\overline{L} : \overline{\langle X \rangle} \to \overline{L}(\overline{\langle X \rangle})$

Then $\phi_{\mathcal{D}} : mod\,\Lambda \to \mathbb{N}$ is defined by $X \mapsto n_{\overline{L}}(\overline{\langle X \rangle})$.

Note: If one takes $\mathcal{D} = \mathcal{P}$, then $\phi_{\mathcal{D}} = \phi$.

Examples

1. Prototypes: $(\phi\dim(\mathcal{D}) = 0)$
   1.1 $\mathcal{D} := \mathrm{Gproj}(\Lambda)$.
   1.2 $\mathcal{D} := {}^{\perp}\Lambda := \{M \in mod\,(\Lambda) \;:\; Ext^i_{\Lambda}(M, \Lambda) = 0 \quad \forall\, i \geq 1\}$.
2. Let T_{Λ} be a tilting module, then we can take as $\mathcal{D}$ the torsion free class $\mathcal{Y}(T) := \{Y \in mod\,\Gamma^{op} : Tor_1^{\Gamma^{op}}(Y, T) = 0\}$.
3. More generally, $\mathcal{D}$ can be any left saturated subcategory.
4. (Trivial?) For any algebra take $\mathcal{D} := mod\,\Lambda$.
   4.1 If Λ is self injective, $\phi\dim(\mathcal{D}) = 0$ and this case is <u>not</u> trivial.
5. Let

$$Q: \quad \alpha \circlearrowright 1 \xrightarrow{\beta} 2 \xrightarrow{\gamma} 3\;; \quad \Lambda = \frac{kQ}{< \alpha^2, \beta\alpha, \gamma\beta >}.$$

$\mathcal{D} := add(\Lambda \oplus S_1 \oplus S_2 \oplus I_2)$ and $\phi\dim(\mathcal{D}) = 2$.

Properties of $\phi_{\mathcal{D}}$

The following properties for $X, Y, M \in mod\,\Lambda$ hold true:

(a) If $M \in \mathcal{D} \cup \mathcal{P}$, then $\phi_{\mathcal{D}}(M) = 0$ and $\phi_{\mathcal{D}}(X \oplus M) = \phi_{\mathcal{D}}(X)$.

(b) $\phi_{\mathcal{D}}(X) \leq \phi_{\mathcal{D}}(X \oplus Y)$.

(c) $\phi_{\mathcal{D}}(X^s) = \phi_{\mathcal{D}}(X)$.

(d) $\phi_{\mathcal{D}}\mathrm{dim}(add\,X) = \phi_{\mathcal{D}}(X)$.

(e) If $pd\,X < \infty$ and $\phi\,\mathrm{dim}(\mathcal{D}) = 0 \Rightarrow \phi_{\mathcal{D}}(X) = \phi(X) = pd\,X$.

Properties of $\phi_{\mathcal{D}}$

Theorem

Let Λ be an artin algebra and $\mathcal{D} \subseteq mod\,\Lambda$ be a class of modules satisfying that $add\,\mathcal{D} = \mathcal{D}$ and $\Omega(\mathcal{D}) \subseteq \mathcal{D}$. Then,

$$\phi(X) \leq \phi_{\mathcal{D}}(X) + \phi\dim(\mathcal{D}),\ \text{for every } X \in mod\,\Lambda.$$

The function $\psi_{\mathcal{D}}$

For any $X \in mod\,\Lambda$, we set

$$\boxed{\psi_{\mathcal{D}}(X) := \phi_{\mathcal{D}}(X) + fin.dim\,(\{Z \in mod\,\Lambda\ :\ Z|\Omega^{\phi_{\mathcal{D}}(X)}(X)\}).}$$

Properties:

(a) Let $Z|\Omega^t(X)$ be such that $0 \le t \le \phi_{\mathcal{D}}(X)$ and $pd\,Z < \infty$. Then $pd\,Z + t \le \psi_{\mathcal{D}}(X)$.

(b) $\psi_{\mathcal{D}}(X) \le \psi_{\mathcal{D}}(X \oplus Y)$.

(c) $\psi_{\mathcal{D}}(X^s) = \psi_{\mathcal{D}}(X)$.

(d) $\psi_{\mathcal{D}}\mathrm{dim}(add\,X) = \psi_{\mathcal{D}}(X)$.

The function $\psi_{\mathcal{D}}$

Theorem

Let Λ be an Artin algebra and $\mathcal{D} \subseteq mod\,\Lambda$ be such that $add\,\mathcal{D} = \mathcal{D}$, $\Omega(\mathcal{D}) \subseteq \mathcal{D}$ and $\phi\dim(\mathcal{D}) = 0$. Then, the following statements hold true.

(a) $\psi(X) \leq \psi_{\mathcal{D}}(X)$, for any $X \in mod\,\Lambda$.

(b) $\psi_{\mathcal{D}}(X \oplus D) = \psi_{\mathcal{D}}(X)$, for any $X \in mod\,\Lambda$ and $D \in \mathcal{D}$.

(c) $\psi_{\mathcal{D}}\dim(\mathcal{D}) = 0$.

Auslander generator:
If $rep.dim(\Lambda) \leq 3$, there exists a module V such that for every $M \in mod\,\Lambda$ there is a short exact sequence:

$$0 \to V_1 \to V_0 \to M \to 0, \text{ with } V_0, V_1 \in add\,V.$$

-K. Igusa, G. Todorov. On the finitistic global dimension conjecture for Artin algebras. Theorem 1.4 and Lemma 1.3 imply that:

$$\text{If } pd\,M < \infty \text{ then } pd\,M \leq \psi(V_0 \oplus V_1) + 1 \leq \psi(V) + 1.$$

Igusa-Todorov algebras:
(J. Wei *Finitistic dimension and Igusa-Todorov algebras*, 2009.)

An Artin algebra Λ is called n-Igusa-Todorov (n-IT-algebra) if there exists a module V and a non negative integer n such that for every $M \in mod\,\Lambda$ there is a short exact sequence:

$$0 \to V_1 \to V_0 \to \Omega^n(M) \to 0, \text{ with } V_0, V_1 \in add\,V.$$

J.Wei proposed the question:
Are all Artin algebras n-IT-algebras for some n??

- The exterior algebra $\Lambda(V)$, when $dim_k V \geq 3$ is **not** IT for any $n \geq 0$. (Teresa Conde PhD thesis 2016.)
- She used a result proved by R. Rouquier in his paper from 2006 "Representation dimension of exterior algebras".

Lat-Igusa-Todorov algebras:
An n-Lat-Igusa-Todorov algebra (n-LIT-algebra, for short), where n is a non-negative integer, is an Artin algebra Λ satisfying the following two conditions:

(a) there is some class $\mathcal{D} \subseteq mod\,\Lambda$ such that $add\,\mathcal{D} = \mathcal{D}$, $\Omega(\mathcal{D}) \subseteq \mathcal{D}$ and $\phi\dim(\mathcal{D}) = 0$;

(b) there is some $V \in mod\,\Lambda$ satisfying that each $M \in mod\,\Lambda$ admits an exact sequence

$$0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow \Omega^n M \longrightarrow 0,$$

such that $X_1 = V_1 \oplus D_1$, $X_0 = V_0 \oplus D_0$, with $V_1, V_0 \in add\,V$ and $D_1, D_0 \in \mathcal{D}$.

Theorem

Let Λ be a n-LIT-algebra. Then

$$fin.dim\,(\Lambda) \leq \psi_{\mathcal{D}}(V) + n + 1 < \infty.$$

We also propose the question:

Are all Artin algebras n-LIT-algebras for some n??