Generalized tilting theory in functor categories

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References

- S. Bazzoni, *The t-structure induced by an n-tilting module*, Trans. Amer. Math. Soc. (to appear).
- R. Martínez-Villa and M. Ortiz-Morales, *Tilting theory and functor categories I. Classical tilting*, Appl. Categ. Struct. **22** (2014), 595–646.
- R. Martínez-Villa and M. Ortiz-Morales, *Tilting theory and functor categories II. Generalized tilting*, Appl. Categ. Struct. **21** (2013), 311–348.

Motivation

- K algebraically closed field
- Λ finite dimensional K-algebra
- T_Λ tilting module
- $\Gamma := \mathrm{End}(T)^{op}$

Brenner-Butler Tilting Theorem

The following statements hold.

(1) $(\mathcal{T}(T), \mathcal{F}(T))$ is a torsion theory, where

$$\mathcal{T}(T) := \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^1_\Lambda(T, M) = 0\},$$

$$\mathcal{F}(T) := \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{Hom}_\Lambda(T, M) = 0\}.$$

Motivation

(2) $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion theory, where

$$\mathcal{X}(T) := \{N \in \operatorname{mod}\Gamma \mid N \otimes_\Gamma T = 0\},$$

$$\mathcal{Y}(T) := \{N \in \operatorname{mod}\Gamma \mid \operatorname{Tor}_1^\Gamma(N, T) = 0\}.$$

(3) There are two category equivalences:

$$\begin{tikzcd}
\mathcal{T}(T) \arrow[drr, leftarrow, "\simeq"] & & \mathcal{F}(T) \arrow[dll, leftarrow, "\simeq"'] \\
\mathcal{X}(T) & & \mathcal{Y}(T).
\end{tikzcd}$$

- R associative ring
- T_R n-tilting module
- $S := \mathrm{End}(T)^{op}$

Miyashita Theorem

There are category equivalences:

$$\mathrm{KE}_e^n(T_R) \underset{\mathrm{Tor}_e^S(-,T)}{\overset{\mathrm{Ext}_R^e(T,-)}{\rightleftarrows}} \mathrm{KT}_e^n({}_ST), \quad \textit{where}$$

$$KE_e^n(T_R) := \{M \mid \mathrm{Ext}_R^i(T,M) = 0, 0 \leqslant i \neq e \leqslant n\},$$

$$KT_e^n({}_ST) := \{N \mid \mathrm{Tor}_i^S(N,T) = 0, 0 \leqslant i \neq e \leqslant n\}.$$

Questions

Observation

$$\mathrm{Mod}(R) \cong \mathsf{Fun}(R, Ab).$$

Replace R with any additive category $\mathcal{C}$, what will happen to the two classical results?

(1) How to define tilting objects in functor categories?

(2) Can we extend Brenner-Butler Theorem to functor categories?

(3) Can we extend Miyashita Theorem to functor categories?

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(1) How to define tilting objects in functor categories?

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(3) Can we extend Miyashita Theorem to functor categories?

Introduction

- $\mathcal{C}$ annuli variety
- $\mathrm{Mod}(\mathcal{C})$:=Fun$(\mathcal{C}^{op}, Ab)$
- $\mathcal{T} \subseteq \mathrm{Mod}(\mathcal{C})$
- $\mathrm{C}(\mathrm{Mod}(\mathcal{C}))$ the category of complexes in $\mathrm{Mod}(\mathcal{C})$
- $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$ the derived category of $\mathrm{Mod}(\mathcal{C})$

Preliminaries

Definition 1.1.1 (Martínez and Ortiz, 2013)

$\mathcal{T}$ is **generalized tilting** if the following hold.

(1) There exists a fixed integer n such that every object T in $\mathcal{T}$ has a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to T \to 0,$$

with each P_i finitely generated.

(2) $\mathrm{Ext}_{\mathcal{C}}^{i\geqslant 1}(T, T') = 0$ for any T and T' in $\mathcal{T}$.

(3) For each $\mathcal{C}(\ \ , C)$, there is an exact resolution

$$0 \to \mathcal{C}(\ \ , C) \to T_C^0 \to \cdots \to T_C^m \to 0,$$

with T_C^i in $\mathcal{T}$.

Definition 1.1.2

$\mathcal{T}$ is **n-tilting** if it is generalized tilting with $\operatorname{pdim}\mathcal{T} \leqslant n$ and for each $\mathcal{C}(\ \ ,C)$, there is an exact resolution

$$0 \to \mathcal{C}(\ \ ,C) \to T_C^0 \to \cdots \to T_C^n \to 0,$$

with T_C^i in $\mathcal{T}$.

Example 1.1.3

Let Λ be an artin R-algebra and let $\mathcal{C} = \operatorname{add}\Lambda$. Assume that T is a classical n-tilting Λ-module. Then $\mathcal{T} = \{\mathcal{C}(\ \ ,M) \mid M \in \operatorname{add}T\}$ is n-tilting.

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Equivalences induced by $\mathcal{T}$

Lemma 2.1.1(Martínez and Ortiz, 2014)

Let's define the following functor:

$$\phi : \mathrm{Mod}(\mathcal{C}) \to \mathrm{Mod}(\mathcal{T}), \phi(M) := \mathrm{Hom}(\quad, M)_{\mathcal{T}}.$$

Then ϕ has a left adjoint:

$$- \otimes \mathcal{T} : \mathrm{Mod}(\mathcal{T}) \to \mathrm{Mod}(\mathcal{C})$$

such that $\mathcal{T}(\quad, T) \otimes \mathcal{T} = T$ for any $T \in \mathcal{T}$.

Theorem 2.1.2

Assume that $\mathcal{T}$ is n-tilting. Then for any $0 \leqslant e \leqslant n$, there are category equivalences

$$\mathrm{KE}_e^n(\mathcal{T}) \underset{\mathrm{Tor}_e^{\mathcal{T}}(\ ,\mathcal{T})}{\overset{\mathrm{Ext}_{\mathcal{C}}^e(\ ,-)_{\mathcal{T}}}{\rightleftarrows}} \mathrm{KT}_e^n(\mathcal{T}), \textit{where}$$

$$KE_e^n(\mathcal{T}) := \{M \mid \mathrm{Ext}_{\mathcal{C}}^i(\ \ ,M)_{\mathcal{T}} = 0, 0 \leqslant i \neq e \leqslant n\},$$

$$KT_e^n(\mathcal{T}) := \{N \mid \mathrm{Tor}_i^{\mathcal{T}}(N,\mathcal{T}) = 0, 0 \leqslant i \neq e \leqslant n\}.$$

- $\mathcal{T}$ generalized tilting with $\operatorname{pdim}(\mathcal{T}) \leqslant n$
- $\mathcal{T}^{\perp_\infty} := \{M \mid \operatorname{Ext}_{\mathcal{C}}^{i\geqslant 1}(T, M) = 0 \ \mathit{for}\ T \in \mathcal{T}\}$

Theorem 2.2.1

The following statements hold.

(1) $({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}), \mathcal{T}^{\perp_\infty})$ is a hereditary and complete cotorsion pair.

(2) $\operatorname{pdim}({}^{\perp_\infty}(\mathcal{T}^{\perp_\infty})) \leqslant n$.

(3) ${}^{\perp_\infty}(\mathcal{T}^{\perp_\infty}) \cap \mathcal{T}^{\perp_\infty} = \operatorname{Add}(\mathcal{T})$.

An isomorphism of Grothendieck groups

Definition 3.1.1 (Martínez and Ortiz, 2013)

- $\mathcal{A} := < |\operatorname{mod}(\mathcal{C})| >$;
- $\mathcal{R} := < [M]-[K]-[L] \mid 0 \to K \to M \to L \to 0 \textit{ is exact in } \operatorname{mod}(\mathcal{C}) >$;

The **Grothendieck group** of $\mathcal{C}$ is $K_0(\mathcal{C}) := \mathcal{A}/\mathcal{R}$.

Theorem 3.1.2

Let $\mathcal{C}$ be an abelian category with enough injectives and $\mathcal{T}$ an n-tilting subcategory of $\operatorname{mod}(\mathcal{C})$ with pseudokernels. Then $K_0(\mathcal{C}) \cong K_0(\mathcal{T})$.

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An abelian model structure

Definition 3.2.1 (J. Gillespie 2004)

Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair on an abelian category $\mathcal{C}$. Let X be a complex.

(1) X is called an $\mathcal{A}$(resp. $\mathcal{B}$) **complex** if it is exact and $Z_n(X) \in \mathcal{A}$(resp. $Z_n(X) \in \mathcal{B}$) for all n.

(2) X is called a dg-$\mathcal{A}$ **complex** if $X_n \in \mathcal{A}$ for each n, and $\mathrm{Hom}(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.

(3) X is called a dg-$\mathcal{B}$ **complex** if $X_n \in \mathcal{B}$ for each n, and $\mathrm{Hom}(A, X)$ is exact whenever A is an $\mathcal{A}$ complex.

Notations

- $\mathcal{T}$ generalized tilting
- $\mathcal{A} := {}^{\perp_\infty}(\mathcal{T}^{\perp_\infty})$
- $\mathcal{B} := \mathcal{T}^{\perp_\infty}$
- $\tilde{\mathcal{A}}$ the class of $\mathcal{A}$ complexes
- $\tilde{\mathcal{B}}$ the class of $\mathcal{B}$ complexes
- $dg\tilde{\mathcal{A}}$ the class of dg-$\mathcal{A}$ complexes
- $dg\tilde{\mathcal{B}}$ the class of dg-$\mathcal{B}$ complexes

Theorem 3.2.2

There is an abelian model structure on $\mathrm{C}(\mathrm{Mod}(\mathcal{C}))$ given as follows:

(1) Weak equivalences are quasi-isomorphisms,

(2) Cofibrations (trivial cofibrations) consist of all the monomorphisms f such that $\mathrm{Coker}\, f \in dg\tilde{\mathcal{A}}(\mathrm{Coker}\, f \in \tilde{\mathcal{A}})$,

(3) Fibrations (trivial fibrations) consist of all the epimorphisms g such that $\mathrm{Ker}\, g \in dg\tilde{\mathcal{B}}(\mathrm{Ker}\, g \in \tilde{\mathcal{B}})$.

The homotopy category of this model category is $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$.

Definition 3.2.3(M. Hovey 2002)

Suppose that an abelian category $\mathcal{A}$ has a model structure and $X \in \mathcal{A}$, X is **trivial** if $0 \to X$ is a weak equivalence, X is **cofibrant** if $0 \to X$ is a cofibration and X is **fibrant** if $X \to 0$ is a fibration.

Corollary 3.2.4

The following statements hold.

(1) X is trivial if and only if X is exact.

(2) C is a cofibrant if and only if $C \in dg\tilde{\mathcal{A}}$.

(3) F is a fibrant if and only if $F \in dg\tilde{\mathcal{B}}$ if and only if F has all the terms in $\mathcal{B}$.

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Suppose that an abelian category $\mathcal{A}$ has a model structure and $X \in \mathcal{A}$, X is **trivial** if $0 \to X$ is a weak equivalence, X is **cofibrant** if $0 \to X$ is a cofibration and X is **fibrant** if $X \to 0$ is a fibration.

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The following statements hold.

(1) X is trivial if and only if X is exact.

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(3) F is a fibrant if and only if $F \in dg\tilde{\mathcal{B}}$ if and only if F has all the terms in $\mathcal{B}$.

A t-structure induced by $\mathcal{T}$

Definition 3.3.1

A **t-structure** on a triangulated category $\mathcal{D}$ is a pair of full subcategories ($\mathcal{X}$,$\mathcal{Y}$) satisfying:

(1) $\mathrm{Hom}_{\mathcal{D}}(X, \Sigma^{-1}Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

(2) $\Sigma\mathcal{X} \subseteq \mathcal{X}$ and $\Sigma^{-1}\mathcal{Y} \subseteq \mathcal{Y}$.

(3) For every object $Z \in \mathcal{D}$ there is a distinguished triangle $X \to Z \to Y \to \Sigma X$ with $X \in \mathcal{X}$ and $Y \in \Sigma^{-1}\mathcal{Y}$.

A t-structure induced by $\mathcal{T}$

- $\mathcal{T}$ generalized tilting
- $k \in \mathbb{Z}$
- $\mathcal{D}^{\leq k}_{\mathcal{T}} = \{X \in \mathrm{D}(\mathrm{Mod}(\mathcal{C})) \mid \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T, X) = 0 \textit{ for any } i < k \textit{ and } T \in \mathcal{T}\}$
- $\mathcal{D}^{\geq k}_{\mathcal{T}} = \{Y \in \mathrm{D}(\mathrm{Mod}(\mathcal{C})) \mid \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}(\mathcal{C}))}(\Sigma^i T, Y) = 0 \textit{ for any } i > k \textit{ and } T \in \mathcal{T}\}$

Proposition 3.3.2

TFAE for $X \in \mathrm{D}(\mathrm{Mod}(\mathcal{C}))$.

(1) $X \in \mathcal{D}_{\mathcal{T}}^{\leq k}$.

(2) $X \cong \cdots \to B_{k+2} \to B_{k+1} \to B_k \to 0 \to \cdots$, with $B_i \in \mathcal{T}^{\perp_\infty}$ for $i \geqslant k$.

(3) $X \cong \cdots \to T_{k+2} \to T_{k+1} \to T_k \to 0 \to \cdots$, with $T_i \in \mathrm{Add}(\mathcal{T})$ for $i \geqslant k$.

Theorem 3.3.3

$(\mathcal{D}_{\mathcal{T}}^{\leq k}, \mathcal{D}_{\mathcal{T}}^{\geq k})$ forms a t-structure on the derived category $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$.

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(1) $X \in \mathcal{D}_{\mathcal{T}}^{\leq k}$.

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$(\mathcal{D}_{\mathcal{T}}^{\leq k}, \mathcal{D}_{\mathcal{T}}^{\geq k})$ forms a t-structure on the derived category $\mathrm{D}(\mathrm{Mod}(\mathcal{C}))$.

Thank you for your attention!