Auslander-Reiten theory in quasi-abelian and Krull-Schmidt categories

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Maurice Auslander Distinguished Lectures
and International Conference 2019

Motivation

GOAL: understand representation theory of partial cluster-tilted algebras

- $\mathcal{C}$ = cluster category (triangulated, Hom-finite, Krull-Schmidt, has a Serre functor)
- Σ = suspension functor
- R = *rigid* object of $\mathcal{C}$, i.e. $\mathrm{Ext}^1_{\mathcal{C}}(R, R) = \mathrm{Hom}_{\mathcal{C}}(R, \Sigma R) = 0$
- $\Lambda_R := (\mathrm{End}_{\mathcal{C}}\, R)^{\mathrm{op}}$ is called a *partial cluster-tilted algebra*

How?

GOAL: to understand $\operatorname{mod}\Lambda_R$

- Use the functor:

$$\mathcal{C} \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(R,-)} \operatorname{mod}\Lambda_R$$

- What happens to the AR theory of $\mathcal{C}$ under $\operatorname{Hom}_{\mathcal{C}}(R,-)$?

Two subcategories

$$\mathcal{X}_R = \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(R, X) = 0\}$$

"kernel of $\operatorname{Hom}_{\mathcal{C}}(R, -)$"

$$\mathcal{C}(R) = \{X \in \mathcal{C} \mid \exists \Delta\colon R_0 \to R_1 \to X \to \Sigma R_0, \text{ some } R_0, R_1 \in \operatorname{add} R\}$$

"R-presented objects"

Cases

A morphism $f: X \to Y$ is *irreducible* if it is neither a section nor a retraction, and $f = hg \Rightarrow g$ is a section or h is a retraction.

$\mathcal{C}(R) =$ "R-presented objects"

1. $X \in \mathcal{C}(R)$ and $Y \in \mathcal{C}(R)$
2. $X \in \mathcal{C}(R)$ and $Y \notin \mathcal{C}(R)$
3. $X \notin \mathcal{C}(R)$ and $Y \in \mathcal{C}(R)$
4. $X \notin \mathcal{C}(R)$ and $Y \notin \mathcal{C}(R)$

The case with few tears: $X \in \mathcal{C}(R)$

Proposition (S.)

Suppose $f\colon X \to Y$ *is irreducible in* $\mathcal{C}$, *where* X, Y *are indecomposable and are not in* $\mathcal{X}_R = \operatorname{Ker}\operatorname{Hom}_{\mathcal{C}}(R, -)$. *Assume* $X \in \mathcal{C}(R)$. *Then*

1. $Y \in \mathcal{C}(R) \Rightarrow \operatorname{Hom}_{\mathcal{C}}(R, f)$ *is irreducible*
2. $Y \notin \mathcal{C}(R) \Rightarrow \operatorname{Hom}_{\mathcal{C}}(R, f)$ *is a section (so not irreducible)*

The case with more tears: $X \notin \mathcal{C}(R)$

What if $X \notin \mathcal{C}(R)$??

Proposition (S.)

Suppose $f\colon X \to Y$ is irreducible in $\mathcal{C}$, where X, Y are indecomposable and are not in $\mathcal{X}_R$. Suppose $X \notin \mathcal{C}(R)$ and $Y \in \mathcal{C}(R)$. If $\overline{f}$ in $\mathcal{C}/[\mathcal{X}_R]$ is right almost split and monic, then $\mathrm{Hom}_{\mathcal{C}}(R, f)$ is irreducible.

The category $\mathcal{C}/[\mathcal{X}_R]$

A *quasi-abelian* category is an additive category with kernels and cokernels in which PBs of cokernels are cokernels and POs of kernels are kernels.

Example

The category of Banach spaces over $\mathbb{R}$

Example

Any torsion class of a torsion pair in an abelian category

Theorem (S.)

$\mathcal{C}/[\mathcal{X}_R]$ is quasi-abelian.

AR theory in quasi-abelian categories

An *AR sequence* in a quasi-abelian category is a short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ where f is minimal left almost split and g is minimal right almost split.

Theorem (S.)

A bunch of AR theory holds in a quasi-abelian category.

Example

Any irreducible morphism is proper monic or proper epic (*or possibly both!*)

AR theory in a quasi-abelian, Krull-Schmidt category

But, $\mathcal{C}/[\mathcal{X}_R]$ is also Krull-Schmidt!

Theorem (S.)

Let $\mathcal{A}$ be a Krull-Schmidt, quasi-abelian category, and $\xi\colon\ X \xrightarrow{f} Y \xrightarrow{g} Z$ an exact sequence in $\mathcal{A}$. Then the following are equivalent.

1. *ξ is an Auslander-Reiten sequence*
2. *$\operatorname{End}_{\mathcal{A}} X$ is local and g is right almost split*
3. *$\operatorname{End}_{\mathcal{A}} Z$ is local and f is left almost split*
4. *f is minimal left almost split*
5. *g is minimal right almost split*
6. *f and g are both irreducible*

Possible future approach

The localisation of an *integral* category at the class of *regular* morphisms gives an abelian category.

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(R,-)} & \operatorname{mod}\Lambda_R \\
\Big\downarrow {\scriptstyle \text{quotient}} & & \\
\mathcal{C}/[\mathcal{X}_R] & \xrightarrow{\text{localisation}} & (\mathcal{C}/[\mathcal{X}_R])[\mathcal{R}^{-1}]
\end{array}
$$

where $\mathcal{R}$ is the class of regular morphisms in $\mathcal{C}/[\mathcal{X}_R]$

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