Auslander-Reiten theory in quasi-abelian and Krull-Schmidt categories

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Motivation

 $\ensuremath{\mathrm{GOAL}}\xspace$ understand representation theory of partial cluster-tilted algebras

- $m{\circ}$ $\mathcal{C}=$ cluster category (triangulated, Hom-finite, Krull-Schmidt, has a Serre functor)
- Σ = suspension functor
- R = rigid object of C, i.e. $\operatorname{Ext}^1_{\mathcal{C}}(R,R) = \operatorname{Hom}_{\mathcal{C}}(R,\Sigma R) = 0$
- $\Lambda_R := (\operatorname{End}_{\mathcal{C}} R)^{\operatorname{op}}$ is called a partial cluster-tilted algebra

How?

GOAL: to understand mod Λ_R

Use the functor:

$$\mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(R,-)} \mathsf{mod}\,\Lambda_R$$

• What happens to the AR theory of $\mathcal C$ under $\mathsf{Hom}_{\mathcal C}(R,-)$?

Two subcategories

$$\mathcal{X}_R = \{X \in \mathcal{C} \mid \mathsf{Hom}_\mathcal{C}(R,X) = 0\}$$
 "kernel of $\mathsf{Hom}_\mathcal{C}(R,-)$ "

$$\mathcal{C}(R) = \{X \in \mathcal{C} \mid \exists \Delta \colon R_0 \to R_1 \to X \to \Sigma R_0, \text{ some } R_0, R_1 \in \mathsf{add} \ R\}$$
 " R -presented objects"

Cases

A morphism $f: X \to Y$ is *irreducible* if it is neither a section nor a retraction, and $f = hg \Rightarrow g$ is a section or h is a retraction.

C(R) = "R-presented objects"

- 2 $X \in \mathcal{C}(R)$ and $Y \notin \mathcal{C}(R)$
- **3** $X \notin \mathcal{C}(R)$ and $Y \in \mathcal{C}(R)$
- \bullet $X \notin \mathcal{C}(R)$ and $Y \notin \mathcal{C}(R)$

The case with few tears: $X \in \mathcal{C}(R)$

Proposition (S.)

Suppose $f: X \to Y$ is irreducible in C, where X, Y are indecomposable and are not in $\mathcal{X}_R = \text{Ker Hom}_{\mathcal{C}}(R, -)$. Assume $X \in \mathcal{C}(R)$. Then

- $Y \in \mathcal{C}(R) \Rightarrow \mathsf{Hom}_{\mathcal{C}}(R,f) \text{ is irreducible }$
- $Y \notin C(R) \Rightarrow \text{Hom}_{C}(R, f)$ is a section (so not irreducible)

The case with more tears: $X \notin \mathcal{C}(R)$

What if $X \notin \mathcal{C}(R)$??

Proposition (S.)

Suppose $f: X \to Y$ is irreducible in C, where X, Y are indecomposable and are not in \mathcal{X}_R . Suppose $X \notin \mathcal{C}(R)$ and $Y \in \mathcal{C}(R)$. If \overline{f} in $\mathcal{C}/[\mathcal{X}_R]$ is right almost split and monic, then $\mathsf{Hom}_{\mathcal{C}}(R,f)$ is irreducible.

The category $\mathcal{C}/[\mathcal{X}_R]$

A *quasi-abelian* category is an additive category with kernels and cokernels in which PBs of cokernels are cokernels and POs of kernels are kernels.

Example

The category of Banach spaces over ${\mathbb R}$

Example

Any torsion class of a torsion pair in an abelian category

Theorem (S.)

 $\mathcal{C}/[\mathcal{X}_R]$ is quasi-abelian.

AR theory in quasi-abelian categories

An AR sequence in a quasi-abelian category is a short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ where f is minimal left almost split and g is minimal right almost split.

Theorem (S.)

A bunch of AR theory holds in a quasi-abelian category.

Example

Any irreducible morphism is proper monic or proper epic (or possibly both!)

AR theory in a quasi-abelian, Krull-Schmidt category

But, $\mathcal{C}/[\mathcal{X}_R]$ is also Krull-Schmidt!

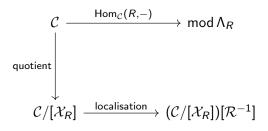
Theorem (S.)

Let \mathcal{A} be a Krull-Schmidt, quasi-abelian category, and $\xi\colon\thinspace X\xrightarrow{f} Y\xrightarrow{g} Z$ an exact sequence in \mathcal{A} . Then the following are equivalent.

- **1** ξ is an Auslander-Reiten sequence
- ② End_A X is local and g is right almost split
- **3** End_A Z is local and f is left almost split
- f is minimal left almost split
- g is minimal right almost split
- of and g are both irreducible

Possible future approach

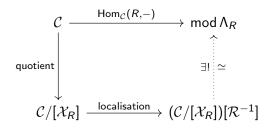
The localisation of an *integral* category at the class of *regular* morphisms gives an abelian category.



where ${\cal R}$ is the class of regular morphisms in ${\cal C}/[{\cal X}_R]$

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