From Cluster Algebras to Quiver Grassmannians

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MetaTheorem/Conjecture

The combinatorics of compatible subsets of maximal Dyck paths controls the geometry of quiver Grassmannians.

Theorem (R.-Weist)

For $k \in \mathbb{Z} \setminus \{1, 2\}$ and $\mathbf{e} = (e_1, e_2) \in \mathbb{Z}^2_{\geq 0}$, the quiver Grassmannian $Gr_{\mathbf{e}}(M_k)$ admits a cell decomposition (affine paving) whose affine cells are naturally labeled by compatible subsets $S \in \mathcal{C}_k$ with

$$|S \cap V_k| = e_2 \qquad |S \cap H_k| = \begin{cases} u_{k-1} - e_1 & \text{if } k \geq 3 \\ u_{1-k} - e_1 & \text{if } k \leq 0 \end{cases}$$

- Fix an integer $n \geq 2$.
- Define *cluster variables* $x_k \in \mathbb{Q}(x_1, x_2)$, $k \in \mathbb{Z}$, recursively by

$$x_{k-1}x_{k+1} = x_k^n + 1.$$

- The first few cluster variables are computed as follows:

$$x_3 = \frac{x_2^n + 1}{x_1}$$

$$x_4 = \frac{x_3^n + 1}{x_2} = \frac{(x_2^n + 1)^n + x_1^n}{x_1^n x_2}$$

$$x_5 = \frac{x_4^n + 1}{x_3} = \frac{N(x_1, x_2)}{x_1^{n^2-1} x_2^n}$$

Here $N(x_1, x_2) \in \mathbb{Z}[x_1, x_2]$ and so a non-trivial cancellation has occurred.

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Theorem (Fomin-Zelevinsky, Laurent Phenomenon)

Each cluster variable $x_k \in \mathbb{Q}(x_1, x_2)$, $k \in \mathbb{Z}$, can be written as

$$x_k = \frac{N_k(x_1, x_2)}{x_1^{d_{k,1}} x_2^{d_{k,2}}},$$

for some polynomial $N_k(x_1, x_2) \in \mathbb{Z}[x_1, x_2]$ with nonzero constant term and some denominator vector $\mathbf{d}_k = (d_{k,1}, d_{k,2}) \in \mathbb{Z}^2$.

First Goal: Understand these Laurent expansions of the cluster variables

- The denominators are relatively easy to describe: define Chebyshev polynomials $u_m = u_m(n) \in \mathbb{Z}$ for $m \in \mathbb{Z}$ recursively by

$$u_1 = 0, \qquad u_2 = 1, \qquad u_{m+1} = nu_m - u_{m-1}.$$

Proposition

For $k \in \mathbb{Z}$, the denominator vector of x_k is given by

$$\mathbf{d}_k = \begin{cases} (u_{k-1}, u_{k-2}) & \textit{if } k \geq 2 \\ (u_{1-k}, u_{2-k}) & \textit{if } k \leq 1 \end{cases}$$

Goal: Understand the numerators $N_k(x_1, x_2)$ of the cluster variables x_k.

I will present two approaches: one geometric and one combinatorial (explaining the relationship between them is the ultimate goal of this talk)

- Let $Q_n = 1 \overset{n}{\longleftarrow} 2$ be the n-*Kronecker quiver* with vertex set $\{1, 2\}$ and arrows α_j, $j = 1, \ldots, n$, from vertex 2 to vertex 1.
- A representation $M = (M_1, M_2, M_{\alpha_j})$ of Q_n consists of the following:
  - $\mathbb{C}$-vector spaces M_i for $i = 1, 2$
  - $\mathbb{C}$-linear maps $M_{\alpha_j} : M_2 \to M_1$ for $j = 1, \ldots, n$
- Write $\underline{\dim}(M) = (\dim M_1, \dim M_2)$ for the *dimension vector* of M
- Given representations $M = (M_1, M_2, M_{\alpha_j})$ and $N = (N_1, N_2, N_{\alpha_j})$, a morphism $\theta : M \to N$ consists of linear maps $\theta_i : M_i \to N_i$ such that $\theta_1 \circ M_{\alpha_j} = N_{\alpha_j} \circ \theta_2$ for $j = 1, \ldots, n$

- A *subrepresentation* $E \subset M$ consists of subspaces $E_i \subset M_i$ such that $M_{\alpha_j}(E_2) \subset E_1$ for $j = 1, \ldots, n$

Definition

Given a dimension vector $\mathbf{e} = (e_1, e_2) \in \mathbb{Z}^2_{\geq 0}$, the *quiver Grassmannian* $Gr_{\mathbf{e}}(M)$ is the set of all subrepresentations $E \subset M$ with $\underline{\dim}(E) = \mathbf{e}$.

Lemma

$Gr_{\mathbf{e}}(M)$ *is a projective variety*

Proof.

$Gr_{\mathbf{e}}(M)$ is naturally identified with a subset of the product of ordinary vector space Grassmannians $Gr_{e_1}(M_1) \times Gr_{e_2}(M_2)$ which is projective. The requirements $M_{\alpha_j}(E_2) \subset E_1$ give closed conditions cutting out the quiver Grassmannian $Gr_{\mathbf{e}}(M)$. □

Theorem (Reineke, Huisgen-Zimmermann, Hille, Ringel)

Every projective variety is isomorphic to a quiver Grassmannian

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Every projective variety is isomorphic to a quiver Grassmannian of Q_n for any $n \geq 3$.

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Moral: one cannot expect great control over the geometry of quiver Grassmannians without imposing conditions on M or $\mathbf{e}$.

- A representation M is *rigid* if $\text{Ext}^1(M, M) = 0$.
- In this case, representations isomorphic to M form a dense subset of the moduli space of representations with dimension vector $\underline{\dim}(M)$.

Theorem (Caldero-Reineke)

Assume $Gr_{\mathbf{e}}(M)$ is nonempty and M is rigid. Then $Gr_{\mathbf{e}}(M)$ is a smooth projective variety.

Theorem (Bernstein-Gelfand-Ponamarev, Dlab-Ringel)

For $k \in \mathbb{Z} \setminus \{1,2\}$, there exists a unique (up to isomorphism) indecomposable rigid representation M_k of Q_n with dimension vector

$$\mathbf{d}_k = \begin{cases} (u_{k-1}, u_{k-2}) & \text{if } k \geq 2 \\ (u_{1-k}, u_{2-k}) & \text{if } k \leq 1 \end{cases}$$

Theorem (Caldero-Chapoton, Caldero-Keller, R./Qin (quantum case))

Each cluster variable $x_k \in \mathbb{Q}(x_1, x_2)$ for $k \in \mathbb{Z} \setminus \{1,2\}$ is a generating function for the Euler characteristics of the quiver Grassmannians for M_k:

$$x_k = \begin{cases} x_1^{-u_{k-1}} x_2^{-u_{k-2}} \sum\limits_{\mathbf{e} \in \mathbb{Z}^2_{\geq 0}} \chi\big(Gr_{\mathbf{e}}(M_k)\big) x_1^{ne_2} x_2^{n(u_{k-1}-e_1)} & \text{if } k \geq 3 \\ x_1^{-u_{1-k}} x_2^{-u_{2-k}} \sum\limits_{\mathbf{e} \in \mathbb{Z}^2_{\geq 0}} \chi\big(Gr_{\mathbf{e}}(M_k)\big) x_1^{ne_2} x_2^{n(u_{1-k}-e_1)} & \text{if } k \leq 0 \end{cases}$$

- Recall the denominator/dimension vectors for $k \in \mathbb{Z}$:

$$\mathbf{d}_k = \begin{cases} (u_{k-1}, u_{k-2}) & \text{if } k \geq 2 \\ (u_{1-k}, u_{2-k}) & \text{if } k \leq 1 \end{cases}$$

Definition

For $k \in \mathbb{Z} \setminus \{1, 2\}$, write D_k for the maximal Dyck path in the lattice rectangle in $\mathbb{Z}^2$ with corner vertices $(0, 0)$ and $\mathbf{d}_k$.

- D_k is a lattice path beginning at $(0, 0)$, taking East and North steps to end at $\mathbf{d}_k$, and never passing above the main diagonal.
- Any lattice point above D_k also lies above the main diagonal.

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For $k \in \mathbb{Z} \setminus \{1, 2\}$, write D_k for the maximal Dyck path in the lattice rectangle in $\mathbb{Z}^2$ with corner vertices $(0, 0)$ and $\mathbf{d}_k$.

- Write H_k and V_k for the sets of horizontal and vertical edges of D_k.
- The edges $H_k \sqcup V_k$ are naturally ordered along the Dyck path D_k from $(0, 0)$ to $\mathbf{d}_k$.

- For $n = 3$, we have

$$\mathbf{d}_3 = (1,0), \qquad \mathbf{d}_4 = (3,1), \qquad \mathbf{d}_5 = (8,3), \qquad \mathbf{d}_6 = (21,8)$$

The associated maximal Dyck paths are shown below:

$D_3 =$


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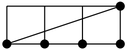

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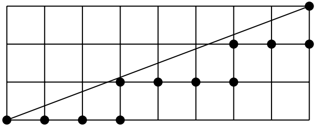

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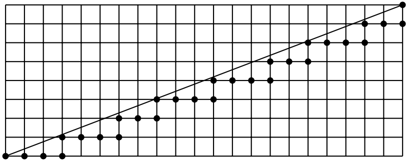

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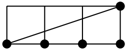
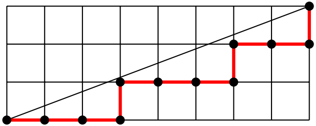
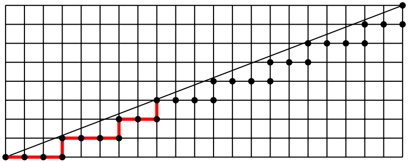
$D_4 =$ 

$D_5 =$ 

$D_6 =$ 

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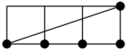
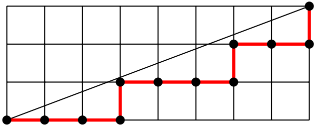
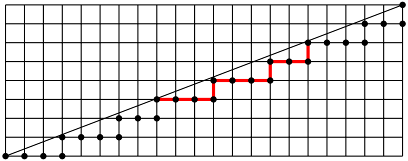
$D_4 =$ 

$D_5 =$ 

$D_6 =$ 

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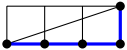
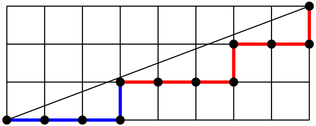
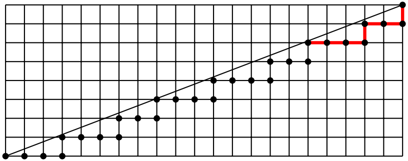
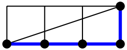
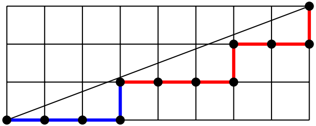
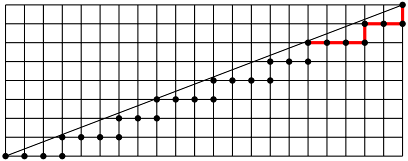
$D_4 =$ 

$D_5 =$ 

$D_6 =$ 

$D_4 =$ 

$D_5 =$ 

$D_6 =$ 

Proposition

For $k \geq 5$, the maximal Dyck path D_k can be constructed by concatenating $n-1$ copies of D_{k-1} followed by a copy of D_{k-1} with its first D_{k-2} removed.

Definition

A subset $S \subset H_k \sqcup V_k$ is *compatible* if for each $h \in S \cap H_k$ and $v \in S \cap V_k$ with $h < v$, there exists an edge $h \leq e \leq v$ such that at least one of the following holds:

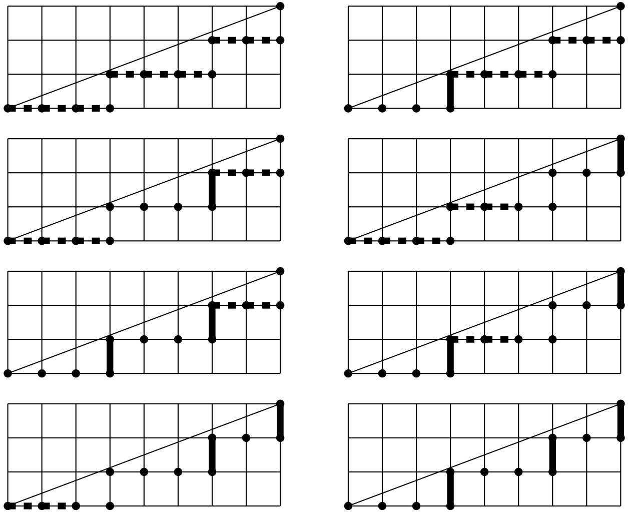
$$e \neq v \qquad \text{and} \qquad n\Big|\{h' \in H_k : h \leq h' \leq e\}\Big| = \Big|\{v' \in V_k : h \leq v' \leq e\}\Big|$$

or

$$e \neq h \qquad \text{and} \qquad n\Big|\{v' \in V_k : e \leq v' \leq v\}\Big| = \Big|\{h' \in H_k : e \leq h' \leq v\}\Big|.$$

Write $\mathcal{C}_k$ for the collection of all compatible subsets of $H_k \sqcup V_k$.

Compatible subsets of D_5 for $n = 3$:



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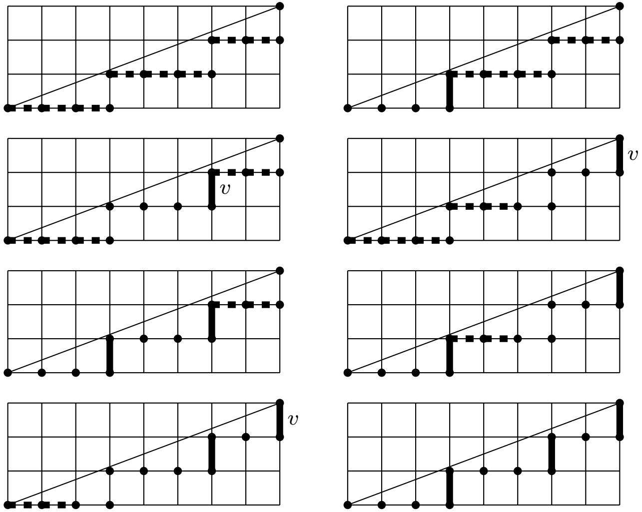

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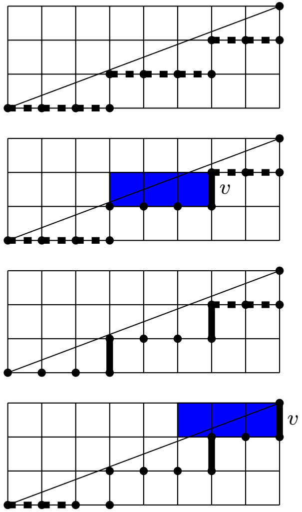

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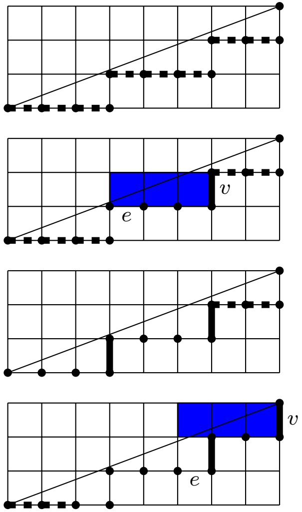

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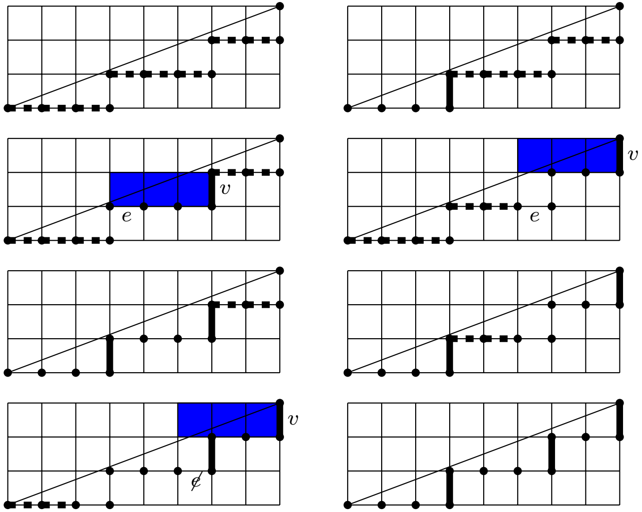

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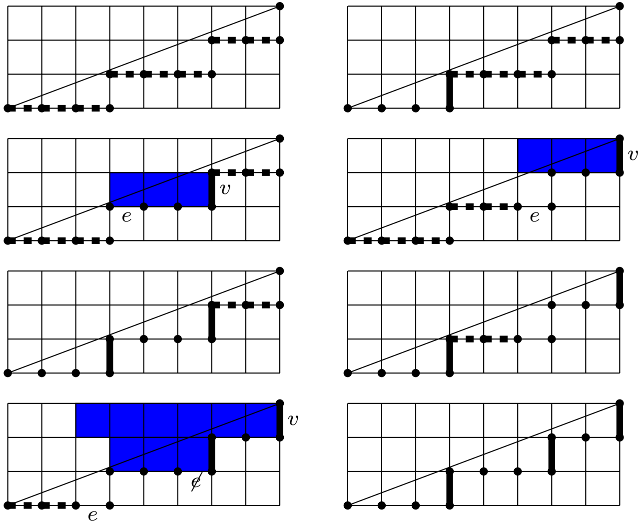

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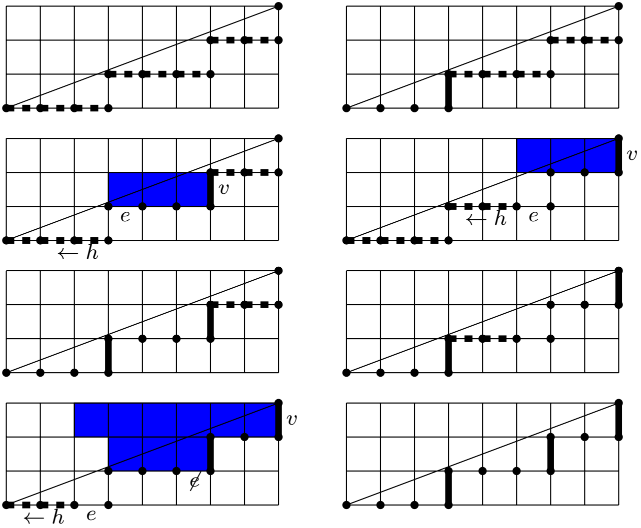


Theorem (Lee-Li-Zelevinsky, R. (quantum/noncommutative case))

For $k \in \mathbb{Z} \setminus \{1, 2\}$, the cluster variable is computed by

$$x_k = x_1^{-d_{k,1}} x_2^{-d_{k,2}} \sum_{S \in \mathcal{C}_k} x_1^{n|S \cap V_k|} x_2^{n|S \cap H_k|}.$$

Corollary

For $k \in \mathbb{Z} \setminus \{1, 2\}$ and $\mathbf{e} = (e_1, e_2) \in \mathbb{Z}^2_{\geq 0}$, the Euler characteristic $\chi\big(Gr_{\mathbf{e}}(M_k)\big)$ is given by the number of compatible subsets $S \in \mathcal{C}_k$ with

$$|S \cap V_k| = e_2 \qquad |S \cap H_k| = \begin{cases} u_{k-1} - e_1 & \text{if } k \geq 3 \\ u_{1-k} - e_1 & \text{if } k \leq 0 \end{cases}$$

Main Question: Why should compatible subsets of D_k "know about" the geometry of $Gr_{\mathbf{e}}(M_k)$?

Theorem (R.-Weist)

For $k \in \mathbb{Z} \setminus \{1, 2\}$ and $\mathbf{e} = (e_1, e_2) \in \mathbb{Z}^2_{\geq 0}$, the quiver Grassmannian $Gr_{\mathbf{e}}(M_k)$ admits a cell decomposition (affine paving) whose affine cells are naturally labeled by compatible subsets $S \in \mathcal{C}_k$ with

$$|S \cap V_k| = e_2 \qquad |S \cap H_k| = \begin{cases} u_{k-1} - e_1 & \text{if } k \geq 3 \\ u_{1-k} - e_1 & \text{if } k \leq 0 \end{cases}$$

- Write $Gr_k(\mathbb{C}^n)$ for the ordinary Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$.
- A subspace $U \in Gr_k(\mathbb{C}^n)$ can be represented by a $k \times n$ matrix by choosing a basis for U.
- Writing such a matrix in reduced row-echelon form gives a unique $k \times n$ matrix representing the subspace U:

$$M(U) := \begin{pmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & 0 \end{pmatrix}$$

- The subset of those $U \in Gr_k(\mathbb{C}^n)$ where $M(U)$ has pivots in columns $\mathbf{a} = \{a_1 < \ldots < a_k\}$ gives the Schubert cell $X_{n,\mathbf{a}} \subset Gr_k(\mathbb{C}^n)$.

- Fix integers $w_1 > \cdots > w_n$ and define a $\mathbb{C}^*$-action on $\mathbb{C}^n$ via $t.e_i = t^{w_i} e_i$, where $\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{C}^n$.
- This induces a $\mathbb{C}^*$ action on $Gr_k(\mathbb{C}^n)$ which can be described on matrix representatives as

$$M(t.U)_{ij} = t^{w_j - w_{a_i}} M(U)_{ij}$$

for $U \in X_{n,\mathbf{a}}$.

Theorem (Białynicki-Birula?)

Each Schubert cell $X_{n,\mathbf{a}}$ is the attractor cell of a $\mathbb{C}^*$-fixed point in $Gr_k(\mathbb{C}^n)$:

$$X_{n,\mathbf{a}} = \left\{U \in Gr_k(\mathbb{C}^n) : \lim_{t \to 0} t.U = \langle e_{a_1}, \ldots, e_{a_k} \rangle \right\}$$

- Let $\mathbb{C}^\ell \subset \mathbb{C}^n$ denote the subspace $\langle e_1, \dots, e_\ell \rangle$. There is an induced short exact sequence

$$0 \longrightarrow \mathbb{C}^\ell \longrightarrow \mathbb{C}^n \longrightarrow \mathbb{C}^{n-\ell} \longrightarrow 0$$

where we identify $\mathbb{C}^{n-\ell}$ with $\langle e_{\ell+1}, \dots, e_n \rangle$.

- This induces a map

$$Gr_k(\mathbb{C}_n) \twoheadrightarrow \bigsqcup_{r+s=k} Gr_r(\mathbb{C}^\ell) \times Gr_s(\mathbb{C}^{n-\ell})$$
$$U \mapsto \big(U \cap \mathbb{C}^\ell, (U + \mathbb{C}^\ell)/\mathbb{C}^\ell\big)$$

which allows to construct the Schubert cells inductively: the preimage of the product of Schubert cells $X_{\ell,\mathbf{b}} \times X_{n-\ell,\mathbf{c}}$ is the Schubert cell $X_{n,(\mathbf{b},\mathbf{c})}$.

Lemma

- *For $k \geq 4$, the space Hom(M_{k-1}, M_k) is n-dimensional.*
- *For any proper subspace $V \subsetneq$ Hom(M_{k-1}, M_k), the natural evaluation map ev : $M_{k-1} \otimes V \to M_k$ is injective.*

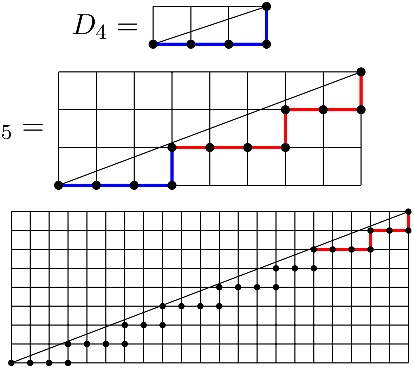
Definition

Define *truncated preprojective* representations $M_k^V := M_k/(M_{k-1} \otimes V)$.

Proposition

For $k \geq 5$ and a codimension-one subspace $V \subsetneq$ Hom(M_{k-1}, M_k), there exists a one-dimensional subspace $\overline{V} \subset$ Hom(M_{k-2}, M_{k-1}) such that $M_k^V = M_{k-1}^{\overline{V}}$.

$D_4 =$ $D_5 =$ $D_6 =$



Proposition

For $k \geq 5$, the maximal Dyck path D_k can be constructed by concatenating $n-1$ copies of D_{k-1} followed by a copy of D_{k-1} with its first D_{k-2} removed.

Theorem (R.-Weist)

For $k \geq 4$ and a proper subspace $V \subsetneq Hom(M_{k-1}, M_k)$ with dim$(V) = r$, each quiver Grassmannian $Gr_{\mathbf{e}}(M_k^V)$ admits a cell decomposition whose affine cells are naturally labeled by compatible subsets of the maximal Dyck path $D_k^{[r]}$ obtained from D_k by removing the first r copies of D_{k-1}.

Idea of Proof

- Caldero-Chapoton maps: given $0 \to A \to M \to B \to 0$ we get

$$Gr_{\mathbf{e}}(M) \to \bigsqcup_{\mathbf{f}+\mathbf{g}=\mathbf{e}} Gr_{\mathbf{f}}(A) \times Gr_{\mathbf{g}}(B)$$

- (iterated) $\mathbb{C}^*$-actions on $Gr_{\mathbf{e}}(M)$ reduce the problem to thinking about quiver Grassmannians on the universal covering quiver of Q_n

Thank you!