# Continuous Clusters and Continuous Mutation

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# Outline

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1. Motivation and Construction
2. Continuous Clusters
3. Continuous Mutation

# Motivation

- Type $A_\infty$ quivers are a infinite-vertex generalizations of type $A_n$ quivers.
- Clusters for type $A_\infty$ quivers are triangulations of the $\infty$-gon, see work by Thorsten Holm and Peter Jørgensen from 2012, [2].
- In 2015, Kiyoshi Igusa and Gordana Todorov introduced a continuous quiver [3].
- They constructed continuous a cluster category whose clusters were discrete maximal geodesic laminations of the hyperbolic plane. (We can think of these as triangulations of the disk, or "$\mathbb{R}$-gon.")
- The construction was limited by "discrete-ness."

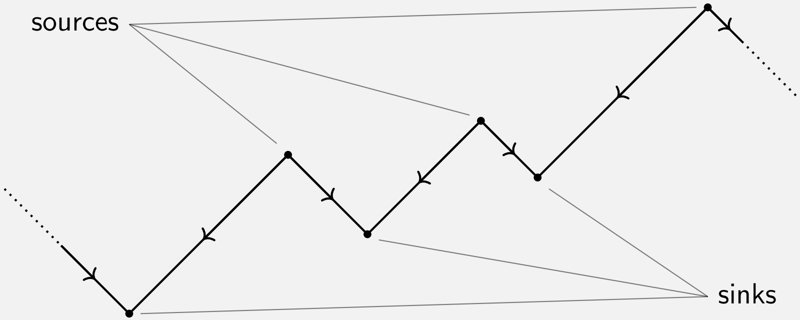
In this talk we have a generalization of clusters and mutation to a continuous setting that behaves well when restricted to previous constructions.

## Construction – $A_{\mathbb{R}}$

### Definition ($A_{\mathbb{R}}$)

We pick a set of sinks and sources in $\mathbb{R}$ with no limit or accumulation points. This induces a partial order $\preceq$ on $\mathbb{R}$. A continuous quiver of type $A$ is $\mathbb{R}$ equipped with such a partial order, denoted $A_{\mathbb{R}}$.

### Example



# Construction – $\text{rep}_k(A_{\mathbb{R}})$

## Proposition (Igusa, R., Todorov)

*The bounded dimensional indecomposables are those whose support is an interval of $\mathbb{R}$, whose linear maps are isomorphisms, and whose vector spaces are $k$ or $0$.*

## Definition ($\text{rep}_k(A_{\mathbb{R}})$)

The category $\text{rep}_k(A_{\mathbb{R}})$ is the category of finite direct sums of one dimensional indecomposables (and usual morphisms of representations).

We denote an indecomposable in $\text{rep}_k(A_{\mathbb{R}})$ by the interval that is its support. For example $[0,1]$ is the indecomposable with support $[0,1]$.

Writing $|$ instead of (, ), [, or ] is short hand for allowing whichever symbol is allowed in context. For example $|0,1|$ can be used for any of $[0,1]$, $[0,1)$, $(0,1]$, or $(0,1)$.

# Construction – $\text{rep}_k(A_{\mathbb{R}})$

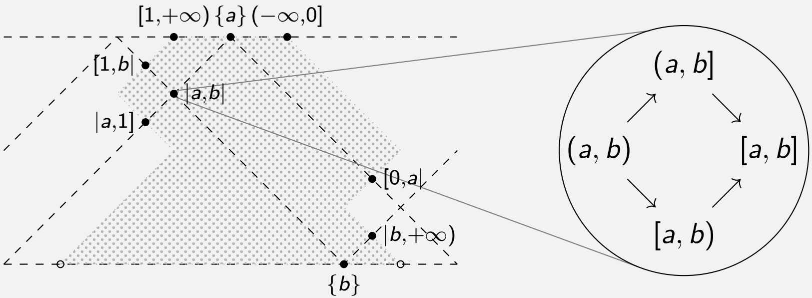
Properties of $\text{rep}_k(A_{\mathbb{R}})$:

- $\text{rep}_k(A_{\mathbb{R}})$ is abelian and Krull-Scmidt. It is not Artinian.
- For indecomposables $V$ and $W$, $\text{Hom}(V, W) = 0$ or $\text{Hom}(V, W) \cong k$.
- For indecomposables $V \ncong W$, either $\text{Hom}(V, W) = 0$ or $\text{Hom}(W, V) = 0$.
- For indecomposables $V$ and $W$, either $\text{Ext}(V, W) = 0$ or $\text{Ext}(V, W) \cong k$.
- For indecomposables $V$ and $W$, either $\text{Ext}(V, W) = 0$ or $\text{Ext}(W, V) = 0$.
- For an indecomposable $V$, $\text{Ext}(V, V) = 0$.
- Most indecomposables belong to a unique Auslander-Reiten sequence. (Picture on next slide.)

# Construction – $\mathrm{rep}_k(A_{\mathbb{R}})$

## Example

Let 0 be a source, 1 a sink, and $a, b \in \mathbb{R}$ such that $0 < a < 1 < b$. Then the *Auslander-Reiten space* looks like this:



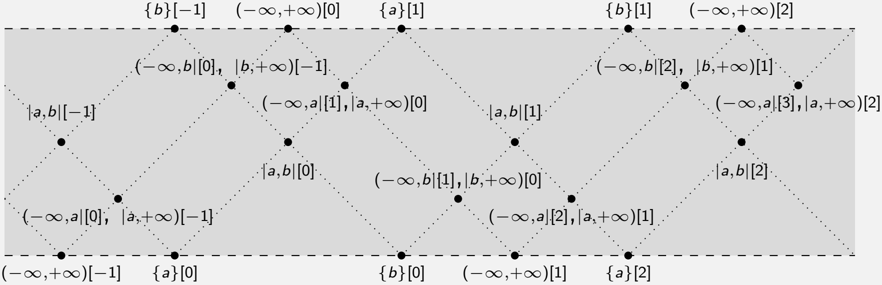

Note: if $A_{\mathbb{R}}$ has the same orientation as $\mathbb{R}$ then the projectives $(-\infty, a|$ of $\mathrm{rep}_k(A_{\mathbb{R}})$ are totally ordered by inclusion just as with $A_n$.

## Definition ($\mathcal{D}^b(A_\mathbb{R})$)

We take $\mathcal{D}^b(A_\mathbb{R})$ to be the usual bounded derived category of the abelian category $\text{rep}_k(A_\mathbb{R})$.

## Example

Take $A_\mathbb{R}$ to have the same ordering as $\mathbb{R}$.



# Construction – $\mathcal{D}^b(A_{\mathbb{R}})$

**Theorem (Igusa, R., Todorov)**

*Let $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ be two different continuous type A quivers. Then $\mathcal{D}^b(A_{\mathbb{R}})$ and $\mathcal{D}^b(A'_{\mathbb{R}})$ are equivalent as triangulated categories if and only if one of the following holds.*

1. *The sinks and sources of both $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ are bounded.*
2. *The sinks and sources of both $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ are each bounded exactly one of above or below.*
3. *The sinks and sources of both $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ are each unbounded above and below.*

In particular, we can now just pretend $A_{\mathbb{R}}$ has the same orientation as $\mathbb{R}$.

# Outline

# Definitions

We do not have an Auslander-Reiten translation defined on all modules, so we instead mod out by shift.

**Definition ($\mathcal{C}(A_{\mathbb{R}})$)**

The continuous cluster category $\mathcal{C}(A_{\mathbb{R}})$ of $A_{\mathbb{R}}$ is the orbit category of $\mathcal{D}^b(A_{\mathbb{R}})$ via [1].

**Corollary (To previous Theorem)**

*If $A_{\mathbb{R}}$ and $A'_{\mathbb{R}}$ are two continuous type A quivers with bounded sinks and sources then $\mathcal{C}(A_{\mathbb{R}})$ is triangulated equivalent to $\mathcal{C}(A'_{\mathbb{R}})$.*

We may continue to pretend $A_{\mathbb{R}}$ has the same ordering as $\mathbb{R}$.

# Definitions

## Definition ($g$-vectors)

Let $K_0^{\text{split}}(\mathcal{C}(A_{\mathbb{R}}))$ denote the split Grothendieck group of $\mathcal{C}(A_{\mathbb{R}})$. A *$g$-vector*, or *index*, of an indecomposable $V$ in $\mathcal{C}(A_{\mathbb{R}})$ is the element $[P_{0,V}] - [P_{1,V}]$ in $K_0^{\text{split}}(\mathcal{C}(A_{\mathbb{R}}))$, where $P_{1,V} \to P_{0,V} \to V$ is the projective resolution of $V$ in $\text{rep}_k(A_{\mathbb{R}})$.

## Definition (Compatible Set, Continuous Cluster)

Let $T$ be a set of indecomposables in $\mathcal{C}(A_{\mathbb{R}})$ such that there is **no pair** $V, W \in T$ with the following nonzero composition:

$$\underbrace{P_{1,V} \to P_{1,W} \to P_{0,V} \to P_{0,W}}_{\neq 0}.$$

(I.e., all $g$-vectors are compatible.) We call $T$ a *compatible set*. If $T$ is maximal with respect to this condition we say $T$ is a *continuous cluster*.

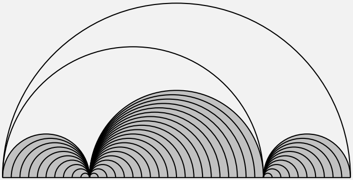
# Definitions

## Example

Let $a < b \in \mathbb{R}$ and

$$\begin{aligned} T = &\{(-\infty, +\infty), (-\infty, a), (a, b), (-\infty, b), (b, +\infty)\} \\ &\cup \{[x, a), \{x\} : -\infty < x < a\} \\ &\cup \{(a, x], \{x\} : a < x < b\} \cup \{(b, x], \{x\} : b < x\}. \end{aligned}$$

$T$ is a continuous cluster with three *continuous fountains*.
We can visualize $T$ as a set of noncrossing partitions of $\mathbb{R}$:

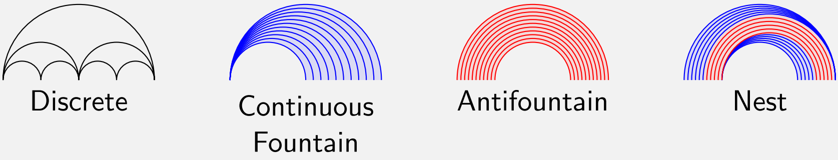


# Structure

The types of compatible sets that make up a continuous cluster are:

- Discrete
- Nests
- Continuous fountains (a type of nest)
- Antifountains (a type of nest)
- Simples

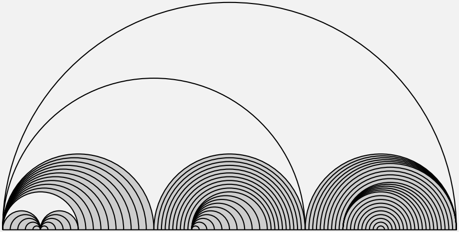
## Example



# Structure

## Example

Here is another picture of a continuous cluster.



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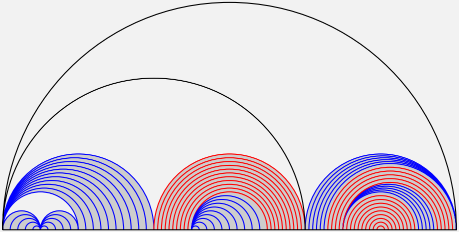


# Structure

The previous construction had a one-to-one correspondence between clusters and discrete maximal geodesic laminations of the hyperbolic plane. We have a nice type of cluster and a similar statement.

## Definition (Laminated Cluster)

Let $T$ be a continuous cluster such that for all $U, V \in T$, if $V \cap U \neq \emptyset$ then

$$(\operatorname{supp} V \cup \operatorname{supp} W) \setminus (\operatorname{supp} V \cap \operatorname{supp} W)$$

is infinite. Additionally, assume that there is at most one indecomposable in $T$ of the form $|a, +\infty)$ where $a \in \mathbb{R}$. We say $T$ is a *laminated cluster*.

## Theorem (Igusa, R., Todorov)

*There is a one-to-one correspondence between laminated clusters and maximal geodesic laminations on the hyperbolic plane.*

# Outline

# Definitions

## Definition (Simple Mutation Path, Mutation Path)

Let $T$ and $T'$ be continuous clusters with respective subsets $S$ and $S'$ where $S \cap S' = \emptyset$; let $\mu : S \to S'$ be a bijection; let $f : S \hookrightarrow [0,1]$ and $g : S' \hookrightarrow [0,1]$ be injections such that

- $g \circ \mu = f$ and $f \circ \mu^{-1} = g$
- $T \setminus S = T' \setminus S'$
- for all $V \in S$, $\{V, \mu V\}$ is not a compatible set
- $(T \setminus f^{-1}([0,t|)) \cup g^{-1}([0,t|)$ is a continuous cluster for all $t \in [0,1]$
- "We can restrict $\mu$, $f$, and $g$ to any subinterval $[0,t|$ and preserve all existing properties."

Then $\mu$, extended to $T$ by $V \mapsto V$ for $V \notin S$ is a *simple mutation path*. A *mutation path* is a finite concatenation of simple mutation paths $\{\mu_i\}$ where the source of $\mu_{i+1}$ is the target of $\mu_i$.

# Definitions

**Theorem (Igusa, R., Todorov)**

*Let $T$ be a continuous cluster and $V \in T$. Let $T_1, T_2$ be continuous clusters, $W_1 \in T_1$ and $W_2 \in T_2$ such that*

$$T \setminus \{V\} = T_1 \setminus \{W_1\} = T_2 \setminus \{W_2\}$$

*and there are simple mutation paths $\mu_1 : T \to T_1$ and $\mu_2 : T \to T_2$. Then $W_1 = W_2$.*

I.e., for any $V \in T$ there is at most one other indecomposable $W$ in $\mathcal{C}(A_{\mathbb{R}})$ such that $\mu : T \to (T \setminus \{V\}) \cup \{W\}$ is a simple mutation path. A simple mutation path performs at most one mutation for every $t \in [0, 1]$, often on an indecomposable that could not previously be mutated.

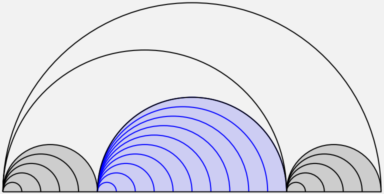
# Examples

## Example

Let $a, b \in \mathbb{R}$ such that $a < 0 < b$. Let $T$ be

$$\{(-\infty, +\infty), (-\infty, a), (a, b), (-\infty, b), (b, +\infty)\}$$
$$\cup \{(-\infty, x], \{x\}, (a, y], \{y\}, (b, z], \{z\}\}$$

for all $x < a < y < b < z$. We're going to "flip" this blue continuous fountain at $a$ to a purple continuous fountain at $b$.



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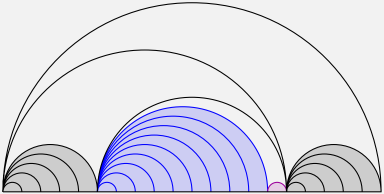

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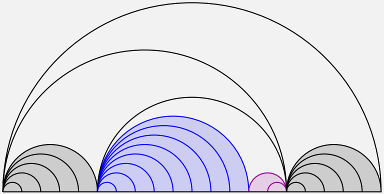

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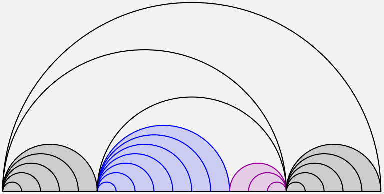

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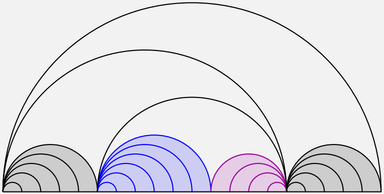

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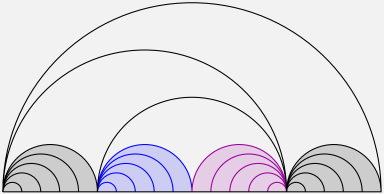

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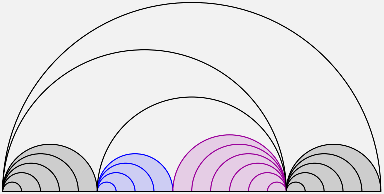

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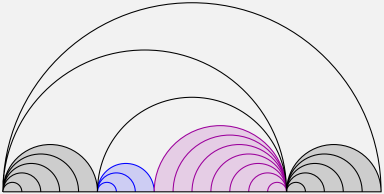

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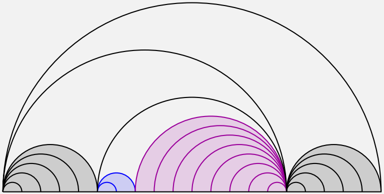

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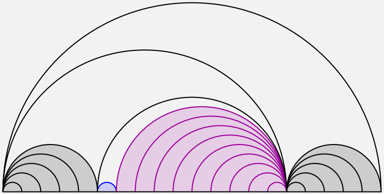

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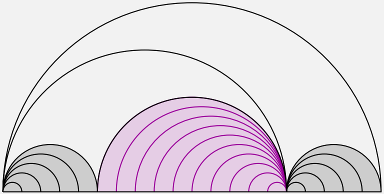

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# Examples

## Example (Lamination)

One reason to use continuous mutation is to *laminate* a cluster. Consider the continuous clusters

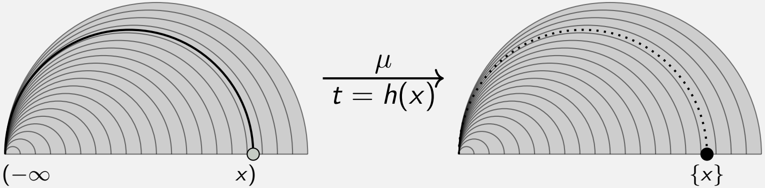
$$T = \{(-\infty, +\infty)\} \cup \{(-\infty, x], (-\infty, x) : x \in \mathbb{R}\}$$
$$T' = \{(-\infty, +\infty)\} \cup \{(-\infty, x], \{x\} : x \in \mathbb{R}\}.$$

$T$ is not laminated but $T'$ is laminated.
Let $S = \{(-\infty, x) : x \in \mathbb{R}\}$ and $\mu : (-\infty, x) \mapsto \{x\}$. Choose any bijection $h : \mathbb{R} \to (0, 1)$ and let $f(-\infty, x) = h(x) = g\{x\}$. Then $\mu : T \to T'$ is a (simple) mutation path. (Picture on next slide.)
We call $T'$ a *lamination* of $T$. A cluster and one of its laminations should essentially contain the same information. The laminated cluster, however, corresponds to a geodesic lamination and is also easier to work with.

# Examples

## Example (Lamination Continued)

Here is a picture:



# Examples

## Example (Transfinite Mutation)

We can also use the same language to encode the transfinite mutation described by Karin Baur and Sira Gratz in 2018, [1]. Let

$$\begin{aligned} T =& \{(n, n+1) : n \in \mathbb{Z}\} \cup \{(-\infty, +\infty)\} \\ & \cup \{(n, n+x], \{x\} : n \in \mathbb{Z}, 0 < x < 1\} \\ & \cup \{(-\infty, n) : n \in \mathbb{Z}\} \\ T' =& \{(n, n+1) : n \in \mathbb{Z}\} \cup \{(-\infty, +\infty)\} \\ & \cup \{(n, n+x], \{x\} : n \in \mathbb{Z}, 0 < x < 1\} \\ & \cup \{(-n, n), (-n, n+1) : n > 0 \in \mathbb{Z}\}. \end{aligned}$$

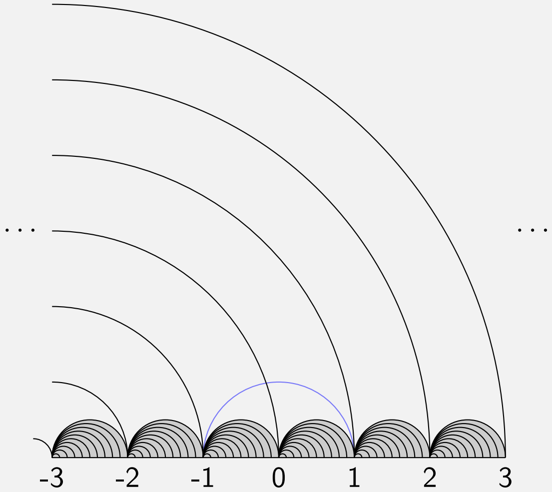
We focus on the last line of each union, which is essentially an $A_\infty$ cluster. The rest just "fills in" the holes between each pair of consecutive integers.

At time $t = 1 - \frac{3}{2^{n+1}}$, we mutate $(-\infty, n) \xrightarrow{\mu} (-n, n+1)$, $n > 0$.

At time $t = 1 - \frac{1}{2^n}$, we mutate $(-\infty, -n) \xrightarrow{\mu} (-n-1, n+1)$, $n \geq 0$.

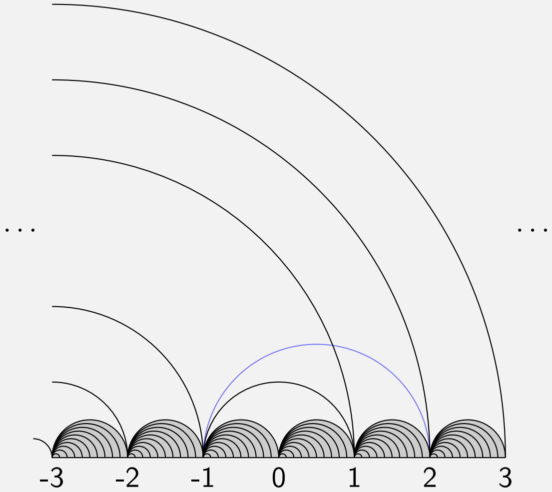
# Examples

## Example (Transfinite Mutation Continued)



# Examples

## Example (Transfinite Mutation Continued)



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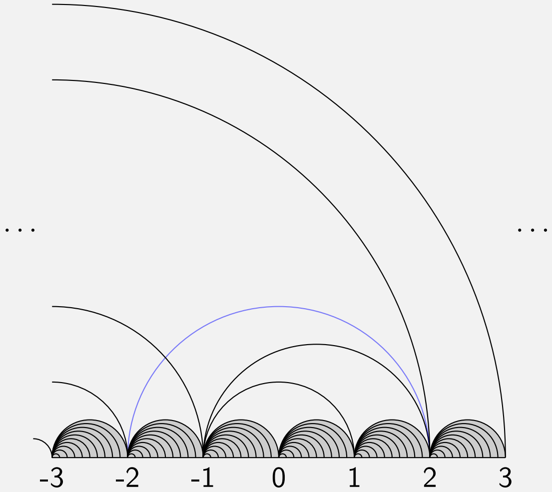

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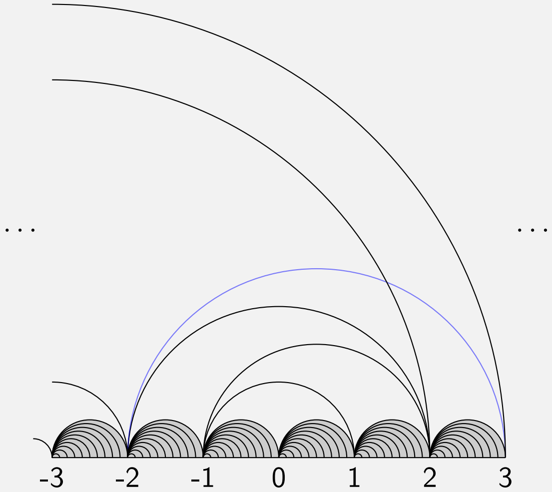

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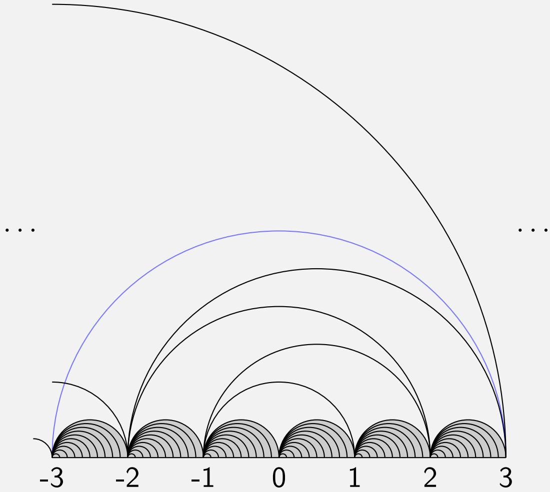
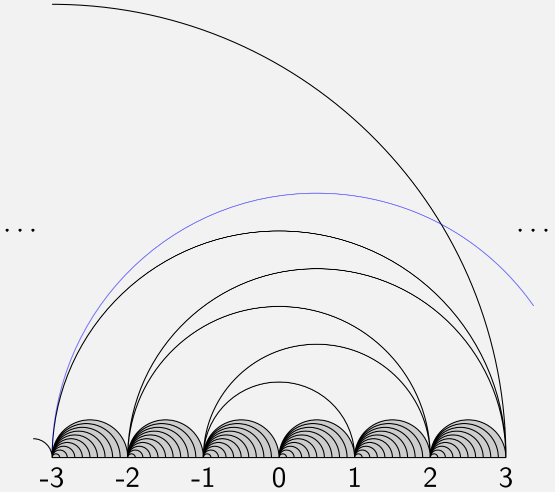


# Examples

## Example (Transfinite Mutation Continued)



# The End

Further Questions:

- Which continuous clusters can be laminated?
- Which continuous clusters are reachable from each other?
- Can we construct a continuous cluster *algebra*?
- Can we use a similar construction (continuous versions of the quiver, clusters, *and* mutation) for type $D$?

# Thank you!

# References


- K. Baur and S. Gratz, *Transfinite mutations in the completed infinity-gon*, Journal of Combinatorial Series A, Volume 155, April 2018, 321 – 359
- T. Holm and P. Jørgensen, *On a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon*, Math. Z. (2012) 270: 277 – 295
- K. Igusa and G. Todorov, *Continuous Cluster Categories I*, Algebras and Representation Theory (2015), 18:65 – 101.