Indecomposable objects determined by their index in Higher Homological Algebra

Joe Reid

The Theorem

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Let $\mathscr{C}$ be a 2d-Calabi-Yau K-linear Hom-finite $(d+2)$-angulated category with split idempotents and d odd, and let $\mathscr{T} = \mathrm{add}(T)$ be an Oppermann-Thomas cluster tilting subcategory.

Assume that if $c, x \in \mathscr{C}$ are indecomposable, then $\mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(c, x)$ and $\mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(x, \Sigma^d(c))$ cannot be simultaneously non-zero.

Then each indecomposable object $c \in \mathscr{C}$ is uniquely determined by its index with respect to $\mathscr{T}$ up to isomorphism.

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Recalling the Index

- We have a $(d+2)$-angulated category $\mathscr{C}$ and an Oppermann-Thomas cluster tilting subcategory $\mathscr{T}$.
- By definition, any object $c \in \mathscr{C}$ appears in a $(d+2)$-angle

$$t_d \to t_{d-1} \to \ldots \to t_0 \to c \to \Sigma^d t_d$$

  where each t_i is an object in $\mathscr{T}$.
- The index of c with respect to $\mathscr{T}$ is defined as

$$\mathrm{Ind}_{\mathscr{T}}(c) := \Sigma_{i=0}^{d}(-1)^i[t_i]$$

  where each $[t_i]$ is an element of $K_0^{\mathrm{split}}(\mathscr{T})$.
- For objects $t \in \mathscr{T}$, we see that $\mathrm{Ind}_{\mathscr{T}}(t) = [t]$

The Triangulated Case

The inspiration came from the following result by Dehy and Keller ("On the combinatorics of rigid objects in 2-Calabi-Yau categories", 2008):

Theorem

Let K be an algebraically closed field, let $\mathscr{C}$ be a K-linear Hom-finite triangulated category with split idempotents, and assume also that $\mathscr{C}$ is 2-Calabi-Yau.

Let $\mathscr{T}$ be a cluster tilting subcategory of $\mathscr{C}$. Then the index induces an injection from the set of isomorphism classes of rigid objects into $K_0^{\mathrm{Split}}(\mathscr{T})$.

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Assume that if $c, x \in \mathscr{C}$ are indecomposable, then $\mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(c, x)$ and $\mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(x, \Sigma^d(c))$ cannot be simultaneously non-zero.

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A Technical Statement

Proposition

Let $\mathscr{C}$ be a K-linear, Hom-finite, $2d$-Calabi-Yau $(d+2)$-angulated category with split idempotents, and let $\mathscr{T}$ be an Oppermann-Thomas cluster tilting subcategory. Suppose that for $c \in \mathscr{C}$ we have a $(d+2)$-angle

$$t_d \to t_{d-1} \to \cdots \to t_1 \to t_0 \to c \to \Sigma^d(t_d)$$

with $t_i \in \mathscr{T}$. Then for any $x \in \mathscr{C}$, there is an exact sequence

$$0 \to \mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(c,x) \to \mathrm{Hom}_{\mathscr{C}}(t_0,x) \to \mathrm{Hom}_{\mathscr{C}}(t_1,x) \to \cdots \to \mathrm{Hom}_{\mathscr{C}}(t_d,x) \to D\mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(x,\Sigma^d(c)) \to 0.$$

The $(d+2)$-angulated case

$$dim_K \mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(c, x) + (-1)^d dim_K \mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(x, \Sigma^d(c)) = \Sigma_{i=0}^{i=d} (-1)^i dim_K \mathrm{Hom}_{\mathscr{C}}(t_i, x).$$

The result holds for indecomposables, and in fact the method used is due to Auslander's work on algebras without short chains.

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Let $\mathscr{C}$ be a $2d$-Calabi-Yau K-linear Hom-finite $(d+2)$-angulated category with split idempotents and d odd, and let $\mathscr{T} = \mathrm{add}(T)$ be an Oppermann-Thomas cluster tilting subcategory.

Assume that if $c, x \in \mathscr{C}$ are indecomposable, then $\mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(c, x)$ and $\mathrm{Hom}_{\frac{\mathscr{C}}{[\Sigma^d \mathscr{T}]}}(x, \Sigma^d(c))$ cannot be simultaneously non-zero.

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The $(d+2)$-angulated higher cluster categories of Dynkin type A

- A family of categories introduced by Oppermann and Thomas ("Higher dimensional cluster combinatorics and representation theory", 2010), which we denote $\mathscr{C}(A_n^d)$.
- These categories are an example of a situation in which the technical assumptions of the result are met.
- A special case of these categories gives the classic triangulated case, as seen in the combinatorial description.

The $(d+2)$-angulated higher cluster categories of Dynkin type A

- To describe $\mathscr{C}(A_n^d)$, take a cyclic ordering of $\mathscr{Z} = \{1, 2, \ldots, n+2d+1\}$; we can consider these the vertices of an $(n+2d+1)$-gon.
- The indecomposables of $\mathscr{C}(A_n^d)$ are in bijection with the subsets of $\mathscr{Z}$ of size $(d+1)$ that contain no neighbouring vertices. We will refer to the indecomposables by their corresponding subset.

The $(d+2)$-angulated higher cluster categories of Dynkin type A

- Two indecomposables X and Y intertwine if there are labellings $X = \{x_0, x_1, \ldots, x_d\}$ and $Y = \{y_0, y_1, \ldots, y_d\}$ such that

$$x_0 < y_0 < x_1 < y_1 < x_2 < \ldots < x_d < y_d < x_0$$

with respect to the ordering.

- The functor Σ^d applied to an indecomposable is equivalent to shifting all of the vertices in the subset down by one.

The $(d+2)$-angulated higher cluster categories of Dynkin type A

- Oppermann-Thomas cluster tilting subcategories can be generated from Oppermann-Thomas cluster tilting objects.
- These are rigid objects with the correct number of indecomposable summands.
- An example of this is the "fan" at a single vertex.

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An Example

- There are several technical assumptions in the result, including the assumption that d is odd.
- Let $\mathscr{C}$ be the 4-angulated higher cluster categories of Dynkin type A_2.
- Let $\mathscr{T}$ be an Oppermann-Thomas cluster tilting subcategory of $\mathscr{C}$.

An Example

Take an indecomposable $t \in \mathscr{T}$, which we recall means that $\mathrm{Ind}_{\mathscr{T}}(t) = [t]$. Then we have the trivial 4-angle

$$t \to t \to 0 \to 0 \to \Sigma^2 t,$$

which gives the 4-angle

$$t \to 0 \to 0 \to \Sigma^2 t \to \Sigma^2 t.$$

This in turn means that $\mathrm{Ind}_{\mathscr{T}}(\Sigma^2 t) = [t]$.

An Example

Let $\mathscr{T}$ be the "fan" at 1; that is, the subcategory generated by all the indecomposable objects of $\mathscr{C}$ containing the vertex 1. Then we see that

$$\mathrm{Ind}_{\mathscr{T}}((1,3,5)) = \mathrm{Ind}_{\mathscr{T}}((2,4,7))$$

which shows that the theorem doesn't generalise to this situation.

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Next Steps

- We are continuing to explore results that can (and cannot) be generalised to the higher dimensional case.
- We are also working on calculating indeces computationally.

Thank you for your attention

- More detail can be found at arXiv:1901.08953
- Any questions?