Matrix Problems Associated to Some Brauer Configuration Algebras
Maurice Auslander Distinguished Lectures Falmouth-USA

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04/25/2019

Road Map

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Aims and Scope

Bijections between solutions of the Kronecker problem and the four subspace problem with indecomposable projective modules over some Brauer configuration algebras are obtained by interpreting elements of some integer sequences as polygons of suitable Brauer configurations.

Ideas from the Medellin CIMPA School

Brauer Configuration Algebras

Definition

Recently, E.L. Green and S. Schroll introduced Brauer configuration algebras as a way to deal with research of algebras of wild representation type (*Brauer configuration algebras: A generalization of Brauer graph algebras, E.L. Green, S. Schroll, Bull. Sci. Math. vol. 141, 2017, 539-572*).

A *Brauer configuration* is a tuple $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ where Γ_0 is a set of vertices, Γ_1 is a set of *polygons*, $\mu : \Gamma_0 \to \mathbb{N}$ is a *multiplicity function* and $\mathcal{O}$ is an orientation, such that the following conditions hold:

- $C(1)$ Every vertex in Γ_0 is a vertex in at least one polygon in Γ_1.
- $C(2)$ Every polygon has at least two vertices.
- $C(3)$ Every polygon has at least a vertex happening more than once (nontruncated vertex).

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The *cyclic ordering* at vertex α is obtained by linearly ordering the list (i.e., $V_{i_1} < \cdots < V_{i_t}$ and by adding $V_{i_t} < V_{i_1}$). Such a list is said to be the *successor sequence* at α.

The Quiver of a Brauer Configuration Algebra

The quiver Q_Γ of a Brauer configuration algebra is defined in such a way that the vertex set $\{v_1, v_2, \ldots, v_m\}$ of Q_Γ is in correspondence with the set of polygons $\{V_1, V_2, \ldots, V_m\}$ in Γ_1, noting that there is one vertex in Q_Γ for every polygon in Γ_1.

Arrows in Q_Γ are defined by the successor sequences.

For each non-truncated vertex $\alpha \in \Gamma_0$ and each successor V' of V at α, there is an arrow from v to v' in Q_Γ where v and v' are the vertices in Q_Γ associated to the polygons V and V' in Γ_1, respectively.

As an example consider a configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ such that:

1. $\Gamma_0 = \{1, 2, 3, 4\}$,
2. $\Gamma_1 = \{U = \{1, 1, 2, 3, 3, 4\}, V = \{1, 2, 3, 4, 4, 4\}\}$,
3. At vertex 1, it holds that; $U < U < V$, $\quad val(1) = 3$,
4. At vertex 2, it holds that; $U < V$, $\quad val(2) = 2$,
5. At vertex 3, it holds that; $U < U < V$, $\quad val(3) = 3$
6. At vertex 4, it holds that; $U < V < V < V$, $\quad val(4) = 4$,
7. $\mu(\alpha) = 1$ for any vertex α.

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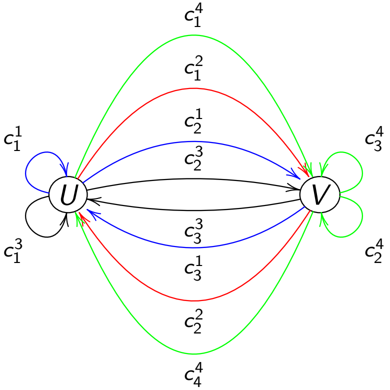
Quiver of the BCA:



(1)

Some Properties of Brauer Configuration Algebras

Definition

Let k be a field and Γ a Brauer configuration. The *Brauer configuration algebra associated to* Γ is defined to be kQ_Γ/I_Γ, where Q_Γ is the quiver associated to Γ and I_Γ is the ideal in kQ_Γ generated by a set of relations ρ_Γ of type I, II and III.

The ideal of relations I_Γ of the Brauer configuration algebra associated to the Brauer configuration Γ is generated by three types of relations:

1. **Relations of type I**. For each polygon $V = \{\alpha_1, \dots, \alpha_m\} \in \Gamma_1$ and each pair of non-truncated vertices α_i and α_j in V, the set of relations ρ_Γ contains all relations of the form $C^{\mu(\alpha_i)} - C'^{\mu(\alpha_j)}$ where C is a special α_i-cycle and C' is a special α_j-cycle.
2. **Relations of type II**. Relations of type II are all paths of the form $C^{\mu(\alpha)}a$ where C is a special α-cycle and a is the first arrow in C.
3. **Relations of type III**. These relations are quadratic monomial relations of the form ab in kQ_Γ where ab is not a subpath of any special cycle unless $a = b$ and a is a loop associated to a vertex of valency 1 and $\mu(\alpha) > 1$.

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Theorem

Let Λ be a Brauer configuration algebra with Brauer configuration Γ.

1. *There is a bijective correspondence between the set of projective indecomposable Λ-modules and the polygons in Γ.*
2. *If P is a projective indecomposable Λ-module corresponding to a polygon V in Γ. Then $\mathrm{rad}\, P$ is a sum of r indecomposable uniserial modules, where r is the number of (non-truncated) vertices of V and where the intersection of any two of the uniserial modules is a simple Λ-module.*
3. *A Brauer configuration algebra is a multiserial algebra.*
4. *The number of summands in the heart of a projective indecomposable Λ-module P such that $\mathrm{rad}^2 P \neq 0$ equals the number of non-truncated vertices of the polygons in Γ*

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Proposition

Let Λ be the Brauer configuration algebra associated to the Brauer configuration Γ. For each $V \in \Gamma_1$ choose a non-truncated vertex α and exactly one special α-cycle C_V at V then

$\{\overline{p} \mid p$ *is a proper prefix of some* $C^{\mu(\alpha)}$ *where* C *is a special* $\alpha-$ *cycle*$\} \bigcup \{\overline{C^{\mu(\alpha)}} \mid V \in \Gamma_1\}$ *is a* k*-basis of* Λ.

Some Matrix Problems

The Kronecker Problem

The classification of indecomposable Kronecker modules was solved by Weierstrass in 1867 for some particular cases and by Kronecker in 1890 for the complex number field case.

This flat matrix problem of type Gelfand is equivalent to the problem of finding canonical Jordan form of pairs (A, B) of matrices with respect to the following elementary transformations:

(i) All elementary transformations on rows of the block matrix (A, B).

(ii) All elementary transformations made simultaneously on columns of A and B having the same index number.

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If k is an algebraically closed field then up to isomorphism every indecomposable Kronecker module belongs to one of the following three classes:

0:

I_n	F_n

where F_n is a Frobenius matrix or companion matrix of a minimal polynomial $p^s(t)$ with $n = s\partial p(t)$, $\partial p(t)$ denotes the degree of the polynomial $p(t)$.

$\mathrm{I} = \mathrm{I}^*$: (a)

I_n	$\mathrm{J}_n(0)$

(b)

$\mathrm{J}_n(0)$	I_n

where $J_n(0) \in \{J_n^+(0), J_n^-(0)\}$ and $J_n^{\pm}(0)$ denotes a corresponding upper or lower Jordan block. Whereas, I^* denotes the dual case defined by the classification problem.

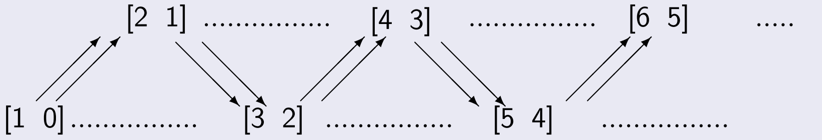
$\mathrm{II} = \mathrm{III}^*$:

$\overrightarrow{\mathrm{I}}_n$	$\overleftarrow{\mathrm{I}}_n$

$\mathrm{III} = \mathrm{II}^*$:

$\mathrm{I}_n^{\uparrow}$	$\mathrm{I}_n^{\downarrow}$

Preprojective Component of the Kronecker Quiver



Helices

Pedro Fernández, Hernán Giraldo and A.M.C associated to each indecomposable preprojective Kronecker module some helices which are paths running through the rows of the matrix block as follows:

$\{a_{1,j}, b_{1,1}, b_{r_1,1}, a_{r_1,s_1}, a_{r_2,s_1}, b_{r_2,s_2}, b_{r_3,s_2}, a_{r_3,s_3}, \dots, l_{r_t,s_t}\}$ where starting vertices are entries in the null row of matrix A.

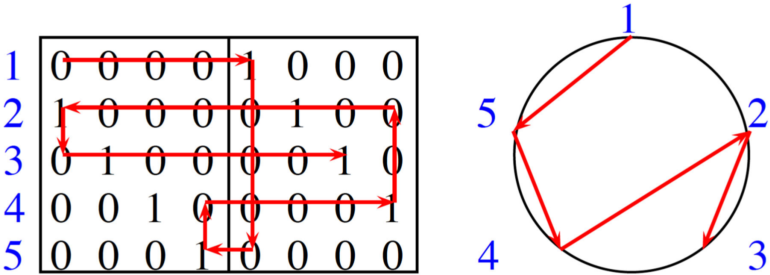


Figure: Preprojective (5, 4); A052558$=\{4, 12, 48, 72, \dots\}$ (the number of helices associated to a preprojective Kronecker module equals the number of ways of connecting $n+1$ equally spaced points on a circle with a path of n line segments ignoring reflections)

Regarding the number of helices associated to preprojective Kronecker modules, we have the following result (Pedro Fernández, Hernán Giraldo, A.M.C)

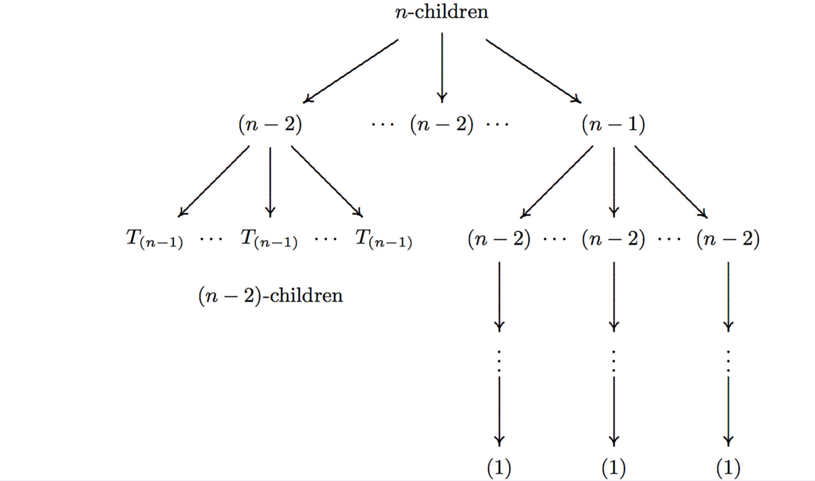
Theorem

If $(n+1, n)$ denotes an indecomposable preprojective Kronecker module then the number of helices associated to $(n+1, n)$ is $h_n^p = n!\lceil \frac{n}{2} \rceil$ where $\lceil x \rceil$ denotes the smallest integer greatest than x. In particular,

$$h_n^p = (n-1)(n-2)h_{n-1}^p + h_{n-1}^i$$

where h_n^i denotes the number of helices associated to the preinjective module $(n, n+1)$.

Proof.



Corollary

For $n \geq 3$ fixed, let Γ be the Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mathcal{O}, \mu)$ such that:

1. $$\begin{aligned} \Gamma_0 &= \{x_1, x_2\}, \\ \Gamma_1 &= \{V_k = x_1^{(2k+2)!} x_2^{(\frac{(k)(2k+2)!}{2})}\}_{1 \leq k \leq n}. \end{aligned} \tag{2}$$

2. *The orientation $\mathcal{O}$ is defined in such a way that for $n \geq 1$*

$$\begin{aligned} &\text{At vertex } x_1;\ V_1^{(4!)} \leq V_2^{(6!)} \leq V_3^{(8!)} \leq \cdots \leq V_n^{((2n+2)!)}, \\ &\text{At vertex } x_2;\ V_1^{(12)} \leq V_2^{(720)} \leq V_3^{(60480)} \leq \cdots \leq V_n^{((\frac{(n)(2n+2)!}{2}))}, \\ &\mu(\alpha) = 1, \quad \textit{for any vertex } \alpha \in \Gamma_0. \end{aligned} \tag{3}$$

3. *the multiplicity function μ is such that $\mu(j) = 1$, for any $j \in \Gamma_0$.*

Then there exists a bijective correspondence between indecomposable projective Λ_Γ-modules and indecomposable preprojective Kronecker modules of the form $(2k+3, 2k+2)$, $1 \leq k \leq n$.

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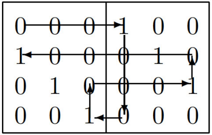
Proof.

The specialization $x_1 = 1$, $x_2 = 2$ makes of each polygon V_k, $k \geq 1$ a unique partition λ of the number

$$h^{p}_{2k+2} = (2k+2)!\lceil k+1 \rceil$$

into parts $\{1, 2\}$ where $\mathrm{occ}(x_i, V_k)$ coincides with the number of times that the part x_i occurs in the corresponding partition since h^{p}_{2k+2} gives the number of helices associated in a unique form to the indecomposable preprojective Kronecker module $(2k+3, 2k+2)$.

$\square$

$(4,3) =$ 
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1 0 0 0 1 0
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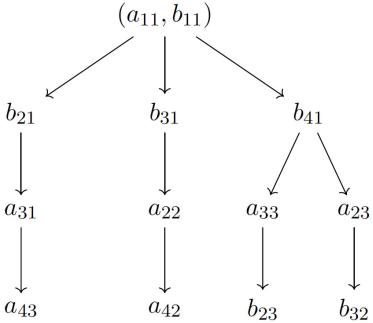
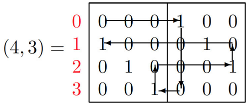
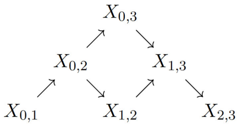

(a_{11}, b_{11})
b_{21}
b_{31}
b_{41}
a_{31}
a_{22}
a_{33}
a_{23}
a_{43}
a_{42}
b_{23}
b_{32}

Helices and Exceptional Sequences

P.F. Fernandez et al proved recently the following result which establishes a relationship between some helices and some exceptional sequences:

Auslander- Reiten quiver of $\mathbb{A}_3$.





$(X_{0,3}, X_{2,3}, X_{1,2})$

For each integers $0 \leq i < j \leq n$ we write X_{ij} the indecomposable whose representations is given by

$$\begin{matrix} 1 & & i & i+1 & & j & j+1 & & n \\ (0 \leftarrow \cdots \leftarrow & & 0 \leftarrow & k & \leftarrow \cdots \leftarrow & k \leftarrow & 0 & \leftarrow \cdots \leftarrow & 0) \end{matrix}$$

Figure: Helices associated to some exceptional sequences. For notation see T. Araya, *Exceptional sequences over path algebras of type* $\mathbb{A}_n$ *and non-crossing spanning trees*, Algebr. Represent. Theory, **16** (1), 239-250,

Theorem

If $(n+1, n)$ denotes an indecomposable preprojective Kronecker module then helices of the form;

$a_{1,1}, b_{1,1}, b_{n+1,1}, a_{n+1,n}, a_{n,n}, b_{n,n}, b_{n-1,n}, a_{n-1,n-2}, \dots, a_{3,2}, a_{2,2}, b_{2,2}$ *when n is even.*

$a_{1,1}, b_{1,1}, b_{n+1,1}, a_{n+1,n}, a_{n,n}, b_{n,n}, b_{n-1,n}, \dots, b_{3,3}, b_{2,3}, a_{2,1}$ *when n is odd.*

correspond to complete exceptional sequences of type $\mathbb{A}_n$.

The Four Subspace Problem

The four subspace problem consists of classifying all indecomposable quadruples (indecomposable representations of four incomparable points as a poset) up to isomorphism.

Zavadskij and Medina gave an elementary solution of this problem (2004).

The Four Subspace Problem (FSP)

Regular Component

$0 = 0^*$

I_n	0	I_n	X
0	I_n	I_n	I_n

$X = F_n(p^s(t)), n \geq 1, \quad p(t) \neq t, t-1$

$I = I^*$

I_n	I_n	0	$J_n^{+}(0)$
0	I_n	I_n	I_n

$II = II^*$

I_n	I_n	0	$J_n^{+}(0)$
0	$I_n^{\leftarrow}$	I_n	I_n

Non − Regular Component

III

I_{n+1}	0	$I_n^{\uparrow}$	$I_n^{\downarrow}$
0	I_n	I_n	I_n

IV

I_{n+1}	0	I_{n+1}	$I_n^{\uparrow}$
0	I_{n+1}	I_{n+1}	$I_n^{\downarrow}$

III^*

I_n	0	$I_n^{\leftarrow}$	$I_n^{\rightarrow}$
0	I_{n+1}	I_{n+1}	I_{n+1}

IV^*

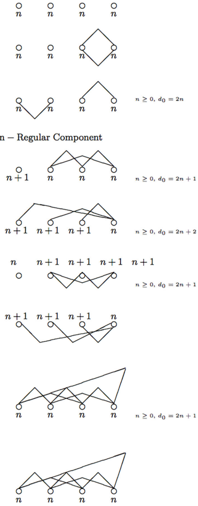
I_{n+1}	0	I_{n+1}	$I_n^{\leftarrow}$
0	I_{n+1}	I_{n+1}	$I_n^{\rightarrow}$

V

I_n	0	$J_n^{+}(0)$	I_n
0	I_n	I_n	$J_n^{+}(0)$
$00\ldots0$	$00\ldots0$	$10\ldots0$	$10\ldots0$

V^*

$I_n^{\leftarrow}$	$I_n^{\leftarrow}$	$I_n^{\rightarrow}$	0
0	$I_n^{\rightarrow}$	$I_n^{\leftarrow}$	$I_n^{\leftarrow}$
$10\ldots0$	$10\ldots0$	$10\ldots0$	$10\ldots0$



The following result establishes a bijection between preprojective representations of type IV and indecomposable projective modules over some Brauer configuration algebras.

Theorem

For $n \geq 2$ fixed, let Γ_n be the Brauer configuration $\Gamma_n = (\Gamma_0, \Gamma_1, \mathcal{O}, \mu)$ such that:

1. $$\begin{aligned}\Gamma_0 &= \{1, 2, 3 \ldots, n, n+1\} \\ \Gamma_1 &= \{V_k\}_{1 \leq k \leq n}, V_i \neq V_j \text{ if } i \neq j.\end{aligned} \tag{4}$$

2. *The orientation $\mathcal{O}$ is defined in such a way that*

   $\mathrm{occ}(1, V_1) = 1$, $\mathrm{occ}(n+1, V_n) = n+1$, *and for* $2 \leq i \leq n$ *at vertex* i, $V_{i-1}^{(i+1,<)} < V_i^{(i^2,<)}$, *where* $V_y^{(x,<)}$ *means that the polygon* V_y *occurs* x *times in the successor sequence of the corresponding vertex,*

3. *the multiplicity function* μ *is such that* $\mu(j) = 1$, *for any* $j \in \Gamma_0$.

Then there exists a bijective correspondence between indecomposable projective Λ_{Γ_n}-modules and indecomposable preprojective representations of type IV *and order* $n \geq 2$ *of the tetrad.*

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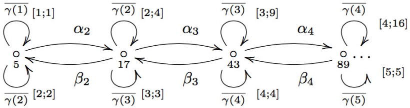
The Four Subspace Problem (FSP)

Firstly, we note that the Brauer configuration (4) allows to see each polygon V_n as a partition of the number h_n into two parts of the form $\{n, n+1\}$ where n occurs $(n)^2$ times and $n+1$ occurs $n+1$ times. Assuming the classical notation for partitions [2] each number h_n can be expressed as follows:

$$h_n = (n)^{(n^2)}(n+1)^{(n+1)}, \quad n \geq 1. \tag{25}$$

we let P_n denote such a partition. The following is the quiver Q_{Γ_n} associated to such Brauer configuration. In this case, we use the symbol $[x; y]$ to denote that the vertex x occurs y times at the corresponding polygon.



(26)

For instance:

$$\begin{aligned}
5 &= (1) + (2+2)\\
17 &= (2+3) + (2+3) + (2+2+3)\\
43 &= (3+3+4) + (3+3+4) + (3+3+4) + (3+3+3+4)\\
89 &= (4+4+4+5) + (4+4+4+5) + (4+4+4+5) + (4+4+4+5) + (4+4+4+4+5)
\end{aligned}$$

Proof.

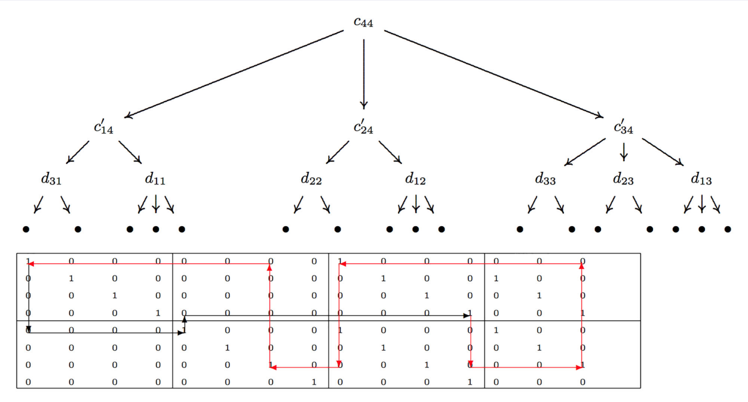


Figure: Partition tree and cycles associated to a preprojective of type IV ($n = 3$). Note that, polygon $17 = (2 + 3) + (2 + 3) + (2 + 2 + 3)$.

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4. A052558, A100705, OEIS (On-Line Encyclopedia of Integer Sequences).
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Thank You