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Supersymmetry and Littlewood-Richardson tableaux

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ABSTRACT. The Littlewood-Richardson coefficients $C_{\mu'\nu}^{\lambda'}$, counting the number of Littlewood-Richardson tableaux of skew shape λ'/μ' and content ν appear in the description of Hook Schur functions. They are characters of simple supermodules over general linear supergroups $G = GL(m|n)$ and the ground field $\mathbb{C}$. The representation theory of $GL(m|n)$ enjoys a natural "supersymmetry," while Littlewood-Richardson tableaux, originating in the representation theory of symmetric groups, lack such a "supersymmetry." The purpose of the talk is to show how to replace Littlewood-Richardson tableaux T by a pair (T^+, T^-) of tableaux related by a "supersymmetry."

We choose a reading P^+ of the entries in the skew-shape diagram

$$\mathcal{D}^+ = \begin{array}{|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & \\ \hline & & \\ \hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

as

$$P^+ = \begin{array}{|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & 6 \\ \hline 1 & 4 & 7 \\ \hline 2 & 5 & 8 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

and to the tableau

$$T^+ = \begin{array}{|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & 4 \\ \hline 4 & 4 & 5 \\ \hline 5 & 6 & 6 \\ \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

we assign the multi-index

$$(I|J) = (11122333|45646456).$$

Analogously, to the tableau

$$T^- = \begin{array}{|c|c|c|} \hline & 3 & 2 & 1 \\ \hline & 3 & 1 & \\ \hline 3 & 2 & 1 & \\ \hline \end{array}$$

and reading

$$P^- = \begin{array}{|c|c|c|} \hline & 3 & 2 & 1 \\ \hline & 5 & 4 & \\ \hline 8 & 7 & 6 & \\ \hline \end{array}$$

we assign the multi-index

$$(K|L) = (12313123|44455666).$$

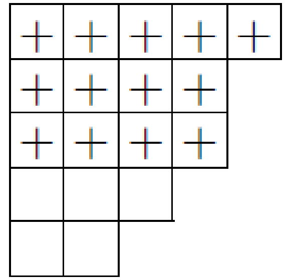
Using supercommutativity, we infer

$$c_{14}c_{15}c_{16}c_{24}c_{26}c_{34}c_{35}c_{36} = c_{I|J} = \pm c_{K|L} = c_{14}c_{24}c_{34}c_{15}c_{25}c_{16}c_{26}c_{36}.$$

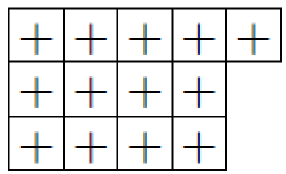
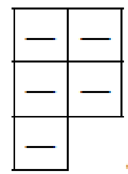
Recall that

$$(I|J) = (11122333|45646456).$$

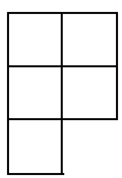
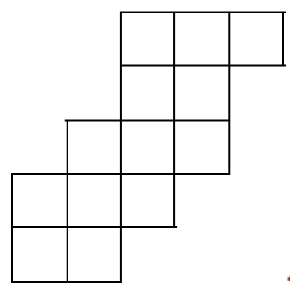
The diagram



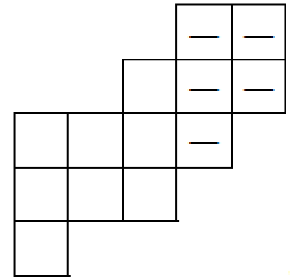
represents a $(3|3)$-hook partition $\lambda = (5,4,4,3,2)$ and corresponds to a weight $(\lambda^+|\lambda^-)$ of the general linear supergroup $GL(3|3)$, where $\lambda^+ = (5,4,4)$ and $\lambda^- = (2,2,1)$ are depicted by diagrams

$\lambda^+ :$  and $\lambda^- :$  .

Consider a partition $\mu = (2,2,1) \lhd \lambda^+$ and the skew shape λ/μ depicted as

$\mu :$  and $\lambda/\mu :$  .

The transpose of λ/μ is the shew shape λ'/μ' depicted as

λ'/μ' :  .

Consider a tableau T of the skew shape λ'/μ' and content $\nu = (5, 4, 4)$ depicted as

$$T = \begin{array}{ccccc} & & & 4 & 4 \\ & & 4 & 5 & 5 \\ 4 & 4 & 5 & 6 & \\ 5 & 6 & 6 & & \\ 6 & & & & \end{array} .$$

By restriction of T, we define the tableau T^+

$$T^+ = \begin{array}{ccc} & & 4 \\ 4 & 4 & 5 \\ 5 & 6 & 6 \\ 6 & & \end{array}$$

of skew shape ${\lambda^+}'/\mu'$ and content ν/λ^-.

To the tableau

$$T^{+} = \begin{array}{|c|c|c|} \cline{3-3} \multicolumn{2}{c|}{} & 4 \\ \hline 4 & 4 & 5 \\ \hline 5 & 6 & 6 \\ \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

of skew shape ${\lambda^{+}}'/\mu'$ and content ν/λ^{-} we assign a tableau

$$T^{-} = \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 3 & 2 & 1 \\ \cline{2-4} \multicolumn{1}{c|}{} & 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

of skew shape ν/λ^{-} and content ${\lambda^{+}}'/\mu'$ by reading the entries of T^{+} from right to left and from top to bottom, and consecutively recording in rows of T^{-} from left to right the column of the entries in T^{+} that equal to a fixed symbol $4, 5$ or 6.

In the above example, the tableau T^{+} is semi-standard, that is, entries in its rows are weakly increasing ($\leq$), and entries in its columns are increasing ($<$).

The tableau T^{-} is anti-semistandard, that is, entries in its rows are decreasing ($>$), and entries in its columns are weakly decreasing ($\geq$).

Proposition 1. *Let* $\lambda = (\lambda^+|\lambda^-)$ *be an* $(m|n)$*-hook partition,* $\mu \lhd \lambda^+$*, and* ν *be a partition such that* $\lambda^- \lhd \nu$*. Let* T *be a tableau of the skew shape* λ'/μ' *and content* ν*. Then* T *is a Littlewood-Richardson tableau if and only if* T^+ *is semistandard and* T^- *is anti-semistandard.*

Moreover, each tableau T^+ *as above that is semi-standard and for which* T^- *is anti-semistandard is the row reading of a picture* $f : {\lambda^+}'/\mu' \to \nu/\lambda^-$ *in the sense of Zelevinski and vice-versa.*

References

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[2] M.: *Even-primitive vectors in induced supermodules for general linear supergroups and costandard supermodules for Schur superalgebras*, J. Algebraic Combin., https://doi.org/10.1007/s10801-019-00879-6, 49pp.

[3] Zelevinsky, A. V.: *A generalization of the Littlewood-Richardson rule and the Richardson-Schensted-Knuth correspondence*, J. Algebra 69 (1981), no. 1, 82–94.