Representations of quantum groups at p^rth root of 1 over p-adic fields

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I. Various representation theories of algebraic groups

The groups

- Let $\mathbf{G}$ be a reductive algebraic group defined over $\mathbb{F}_q$ and $\mathbf{k} = \bar{\mathbb{F}}_q$.

Example: $\mathbf{G}L_n$ is defined over $\mathbb{Z}$. For any commutative ring A, $\mathbf{G}L_n(A)$ is the group of all invertible matrices in with entries in A.

Ring homomorphism $f : A \to B$ gives a group homomorphism

$$\mathbf{G}L_n(f) : \mathbf{G}L_n(A) \to \mathbf{G}L_n(B).$$

• There are many groups associated to $\mathbf{G}$ by taking rational points over various fields:

– Finite groups $G(q^r) = \mathbf{G}(\mathbb{F}_{q^r})$

– Infinite groups $G = \mathbf{G}(\mathbf{k})$ for any field extension $\mathbf{k} \supseteq \mathbb{F}_q$

– The groups $\mathbf{G}(\mathbb{F}_q[t]/t^n)$ and the limit $\mathbf{G}(\mathbb{F}_q[[t]]) \subseteq \mathbf{G}(\mathbb{F}_q((t)$

– The groups $\mathbf{G}(\bar{\mathbb{F}}_q[t]/t^n)$ and the limit $\mathbf{G}(\bar{\mathbb{F}}_q[[t]]) \subseteq \mathbf{G}(\bar{\mathbb{F}}_q((t)$

– p-adic groups $\mathbf{G}(\mathbb{Q}_p)$

• Profinite groups and proalgebraic groups Consider smooth representations.

• Representation theory of $G(q^r)$ over a field $\mathbf{K}$: The classical question: for characteristics of $\mathbf{K}$ being the same as that of $\mathbb{F}_q$ or different.

- Rational representation theory of $\mathbf{G}$ (representations over $\mathbf{k}$), one of the main topics.

- Representations of the infinite groups $G = \mathbf{G}(\mathbf{k})$ as an abstract group over a field $\mathbf{K}$

- Representations of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$ (over the defining field $\mathbf{k}$), both restricted representations and other representations.

Example: For $\mathbf{G} = \mathbf{GL}_n$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{k}) = \mathrm{End}_{\mathbf{k}}(\mathbf{k}^n)$. The restricted structure is the map $x \mapsto x^p \in \mathrm{End}_{\mathbf{k}}(\mathbf{k}^n)$.

- Representations of the Frobenius kernels $\mathbf{G}_r$ and their thickenings.

Example: For $\mathbf{G} = \mathbf{GL}_n$, $\mathbf{G}_r(A) = \ker(Fr : G(A) \to G(A))$ with $Fr((a_{ij}) = (a_{ij}^q)$.

• Representations of the hyperalgebra (or distribution algebra) $D(\mathbf{G}) = \text{Dist}(\mathbf{G})$ and its finite dimensional subalgebras $D_r(\mathbf{G}) = \text{Dist}(\mathbf{G}_r)$.

Example: For $\mathbf{G} = \mathbf{G}_a$,

$$\text{Dist}(\mathbf{G}) = \mathbf{k}\text{-span}\{x^{(n)} \mid n \in \mathbb{N}\}/\sim$$

$$x^{(n)}x^{(m)} = \binom{n+m}{n} x^{(n+m)}$$

"think of" $x^{(n)} = x^n/n!$

$\text{Dist}(\mathbf{G}_r) = \mathbf{k}\text{-span}\{x^{(n)} \mid n < q^r\}$

Example: For $\mathbf{G} = \mathbf{G}_m$,

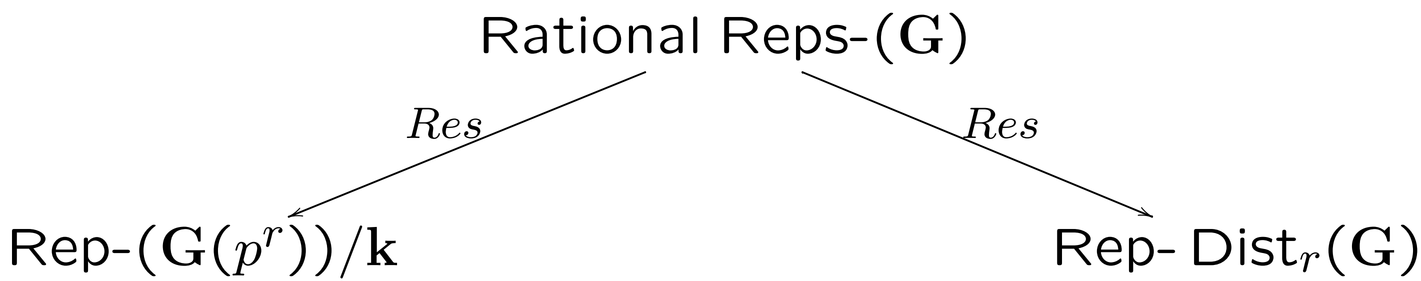
$$\text{Dist}(\mathbf{G}) = \mathbf{k}\text{-span}\{\delta_{(n)} \mid n \in \mathbb{N}\}$$

$$\delta_{(n)}\delta_{(m)} = \sum_{i \geq 0} \binom{n+m-i}{n-i, m-i, i} \delta_{(n+m-i)}$$

"think of" $\delta_{(n)} = \binom{\delta_1}{n}$

$\text{Dist}(\mathbf{G}_r) = \mathbf{k}\text{-span}\{\delta_{(n)} \mid n < q^r\}$.

Relations



- Relations among these representation theories are complicated. Some of them have quantum analog and others, not known yet.

- Representations of $G(q^r)$ over $\mathbf{k}$ and that of $D_r(\mathbf{G})$ and $\mathbf{G}_r$, and rational representations are well studied. Irreducibles, projectives, cohomology theories etc.

- Representations of $G(q^r)$ over $\mathbb{C}$, or $\bar{\mathbb{Q}}_l$ ($l \neq p$)for all r. Character theory controls everything: How to compute the characters? directly compute, one group at a time. Deligne-Lusztig characters, and Lusztig's character sheaf theory: certain perverse sheaves on the algebraic variety $\mathbf{G}(\mathbf{k})$ (constructible l-adic sheaves with values in $\bar{\mathbb{Q}}_l$.

• Representations of $G(q^r)$ and over $\mathbf{K} = \bar{\mathbf{K}}$ with $\mathrm{ch}(\mathbf{K}) \neq \mathrm{ch}(\mathbb{F}_q)$, there are also geometric approach by considering the constructible sheaves with coefficient in $\mathbf{K}$ by Juteau and many others using Langland dual group.

Theorem 1 (Borel-Tits-1973). *Let $\mathbf{G}$ and $\mathbf{G}'$ be two simple algebraic groups over two different fields $\mathbf{k}$ and $\mathbf{k}'$ respectively. If there is an abstract group homomorphism $\alpha : \mathbf{G}(\mathbf{k}) \to \mathbf{G}'(\mathbf{k}')$ such that $\alpha([\mathbf{G}, \mathbf{G}])$ is dense in $\mathbf{G}'(\mathbf{k}')$, then α "almost" rational algebraic group homomorphism. In particular there is field homomorphism $\mathbf{k} \to \mathbf{k}'$ and $\mathrm{char}(\mathbf{k}) = \mathrm{char}(\mathbf{k}')$.*

Essentially if $\mathbb{E}$ and $\mathbf{k}$ have different characteristic, the infinite group $\mathbf{G}(\mathbf{k})$ does not have finite dimensional non-trivial representations.

Example 1. Let $\mathbf{G} = \mathbb{G}_m = \mathrm{GL}_1$ be the multiplicative group scheme. $\mathbf{G}(\mathbf{k}) = \mathbf{k}^\times$.

$W_p(\mathbf{k})$ — the ring of Witt vectors of the field $\mathbf{k}$.

$\mathbf{K}$ — the field of fractions of $W_p(\mathbf{k})$.

Then the commutative group $\mathbb{G}_m(\mathbf{k})$ has plenty one dimensional representations. For example, the Teichmüller representative $\tau : \mathbf{k}^\times \to W_p(\mathbf{k})^\times \subset \mathbf{G}L_1(\mathbf{K})$ is a group character. The Galois groups $\mathrm{Gal}(\mathbf{k})$ acts on the set of all characters.

Remark: $W_p(\mathbb{F}_p) = \mathbb{Z}_p$, the p-adic integers, $\mathbf{K} = \mathbb{Q}_p$.

More general

$$\mathrm{Det} : \mathbf{G}L_n(\mathbf{k}) \to \mathbf{k}^\times \xrightarrow{\tau} W_p(\mathbf{k})^\times \subset \mathbf{G}L_1(\mathbf{K}).$$

Example 2. $\mathbf{G} = \mathbb{G}_a$, $\mathbb{G}_a(\mathbf{k}) = (\mathbf{k}, +)$. Fix any pth root $\xi \in \mathbf{K}$ of 1, $\psi : \mathbb{Z}/p\mathbb{Z} \to \mu_p \subseteq \mathbf{K}^\times$ by $\psi(n) = \xi^n$. $\mathbf{k}$ is a $\mathbb{F}_p$ vector space and choose a basis, one has non-countablely many irreducible representations if $\mathrm{Ch}(\mathbf{K}) \neq p$ and one single irreducible representation if $\mathrm{Ch}(\mathbf{K}) = p$.

Remark 1. $\mathbf{G}(\mathbf{k}) = \cup_{r \geq 1} G(q^r)$ is a union of finite groups.

Reductive groups are built up from $\mathbf{G}_m$'s and $\mathbf{G}_a$'s through the root systems.

There are subgroups $\mathbf{G} \supset \mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$ and $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ all defined over $\mathbb{F}_q$ and they have corresponding subgroups of rational points.

• The representations of the infinite group $\mathbf{G}(\mathbf{k})$ were considered by Nanhua Xi in 2011 using the fact that $\mathbf{G}(\mathbf{k})$ is a directed union of finite groups of Lie type.

The standard constructions of induced representations and Harish-Chandra induced representations have interesting decompositions (with finite length). But induced modules are no longer semisimple (even over $\mathbb{C}$) and the Hecke algebras are trivial.

Example The induced module $\mathbf{KG}(\bar{\mathbb{F}}_p) \otimes_{\mathbf{KB}(\bar{\mathbb{F}}_p)} \mathbf{K}$ has only finitely many composition factors indexed by subsets of simple roots and each appears exactly once in all characteristics. But $\text{End}(\mathbf{KG}(\bar{\mathbb{F}}_p) \otimes_{\mathbf{KB}(\bar{\mathbb{F}}_p)} \mathbf{K}) = \mathbf{K}$. The Hecke algebra is trivial even for $\mathbf{K} = \mathbb{C}$.

• When $\mathbf{K} = \mathbf{k}$, then both finite dimensional representations (rational representations) and non-rational representations (infinite dimensional representations) all appear.

Remark 2. $D(\mathbf{G}) = \cup_{r \geq 1} D_r(\mathbf{G})$ is also a union of finite dimensional Hopf subalgebras.

The goal is to relate representations of $D(\mathbf{G})$ and that $\mathbf{G}(\mathbf{k})$ over $\mathbf{k}$, in terms of Harish-Chandra inductions. The best analog is the category $\mathcal{O}$ of the Hyperalgebra $D(\mathbf{G})$.

II. Irreducible characters in category $\mathcal{O}$

Let $U = \mathrm{Dist}(\mathbf{G})$ Then $U = U^- \otimes_{\mathbf{k}} U^0 \otimes_{\mathbf{k}} U^+$, as $\mathbf{k}$-vector space.

The commutative and cocommutative Hopf $\mathbf{k}$-algebra $U^0 = \bigotimes \mathrm{Dist}(\mathbb{G}_m)$ (not finitely generated) defines an abelian group scheme $X = \mathrm{Spec}(U^0)$ with group operation written additively. Let $X(\mathbf{k})$ denote the $\mathbf{k}$-rational points of X.

Kostant $\mathbb{Z}$-form defines a $\mathbb{Z}$ structure on X and $X(\mathbf{K}) = (\mathfrak{h}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbf{K})^*$ if $\mathrm{char}(\mathbf{K}) = 0$ and $X(\mathbf{k}) = X(W_p(\mathbf{k})) \supseteq X(\mathbb{Z}_p)$.

$X(\mathbf{k}) = X(W_p(\mathbf{k}))$ is a free $W_p(\mathbf{k})$-module with a basis $\{\omega_i\}$ (the fundamental weights).

If $Q = \mathbb{Z}\Phi$ is the root lattice, then there is a paring $Q \times X(\mathbf{k}) \to W_p(\mathbf{k})$ with $(\alpha, \lambda) = \langle \alpha^\vee, \lambda \rangle$.

$$0 \to p^r X(\mathbf{k}) \to X(\mathbf{k}) \to X_r \to 0$$

- Verma modules $M(\lambda) = U \otimes_{U^{\geq 0}} \mathbf{k}_\lambda$ with $\lambda \in X(\mathbf{k})$.

- $M(\lambda)$ has unique irreducible quotient $L(\lambda)$.

Inductive limit property:

- $M(\lambda) = \cup_{r=1}^{\infty} \mathsf{Dist}(\mathbf{G}_r) v_\lambda^+$.

- $L(\lambda) = \cup_{r=1}^{\infty} \mathsf{Dist}(\mathbf{G}_r) v_\lambda^+$.

- Each module M in the category $\mathcal{O}$ defines function $\mathsf{ch}_M : X(\mathbf{k}) \to \mathbb{N}$, written as formal series:

$$\mathsf{ch}_M = \sum_{\lambda \in X(\mathbf{k})} \mathsf{dim}(M_\lambda) e^\lambda.$$

- One has to replace group algebra $\mathbb{Z}[X(\mathbf{k})]$ by function algebra with convex conical supports on $X(\mathbf{k})$ in order for convolution product to make sense.

• Frobenius morphism $Fr : \mathbf{G} \to \mathbf{G}$ over $\mathbb{F}_q$ defines a map $X(\mathbf{k}) \to X(\mathbf{k})$ ($\lambda \mapsto \lambda^{(1)} = q\lambda$). Similarly $\lambda^{(r)} = q^r\lambda$ Frobenius twisted representation.

Theorem 2 (Haboush 1980). *For each* $\lambda = \sum_{r=0}^{\infty} p^r \lambda^r \in X(\mathbf{k})$,

$$L(\lambda) = L(\lambda^0) \otimes L(\lambda^1)^{(1)} \otimes L(\lambda^2)^{(2)} \otimes \cdots$$

Infinite tensor product should be understood as direct limit.

Goal: compute the character $\mathrm{ch}_{L(\lambda)}$ in terms of the function $\mathrm{ch}_{M(\mu)}$.

Haboush theorem implies

$$\text{ch}_\lambda = \prod_{r=1}^{\infty} (\text{ch}_{L(\lambda^r)})^{(r)}.$$

The infinite product makes sense in the function spaces.

Example 3. Let $\lambda = -\rho \in X(\mathbb{Z}) \subseteq X(\mathbb{Z}_p) = X(\mathbf{k})$. Then

$$L(-\rho) = M(-\rho) = L((q-1)\rho) \otimes L((q-1)\rho)^{(1)} \otimes L((q-1)\rho)^{(r)} \otimes$$

using the fact $-1 = \sum_{r=0}^{\infty} (q-1)q^r$.

III. Generic quantum groups over a p-adic field–Nonintegral weights

- Let $\mathbb{Q}_p' = \mathbb{Q}_p[\xi]$ where ξ is a p^r-th root of 1.

- $\mathbb{Q}_p'$ is a discrete valuation field and let $\mathbb{A}$ be the ring of integers in $\mathbb{Q}_p'$ over $\mathbb{Z}_p$. Then $\mathbb{A}$ is a complete discrete valuation ring with maximal ideal $p\mathbb{A}$ generated by p.

- Each $\lambda \in \mathbb{Z}_p$ defines a $\mathbb{Q}_p'$ algebra homomorphism $\mathbb{Q}_p'[K, K^{-1}] \to \mathbb{Q}_p'$ by sending $K \to \xi^\lambda$.

- $\xi^\lambda \in \mathbb{A}$. In fact $\xi \in \mathbb{Q}_p'$ is a p^rth-root of 1 implies $z = \xi - 1 \in p\mathbb{A}$ and

$$(1+z)^\lambda = \sum_{n=0}^{\infty} \binom{\lambda}{n} z^n \text{ converges in } \mathbb{Q}_p',\ \forall \lambda \in \mathbb{A}.$$

- For an indeterminate v, set $z = v - 1 \in \mathbb{Z}[v, v^{-1}]$. $v^{-1} = \sum_{n=0}^{\infty}(-1)^n z^n \in \mathbb{A}[[z]]$ implies $\mathbb{Z}[v, v^{-1}] \subseteq \mathbb{A}[[z]]$ and $\mathbb{Q}(v) \subseteq \mathbb{Q}'_p((z))$. For any $\lambda \in \mathbb{Z}_p[[z]]$

$$v^{\lambda} = \sum_{n=0}^{\infty} \binom{\lambda}{n} z^n$$

is convergent in $\mathbb{Z}_p[[z]]$ by noting that $\binom{\lambda}{n} \in \mathbb{Z}_p[[z]]$.

- Let $\mathbf{U}_{\mathbb{C}(v)}$ (generic case) be the quantum enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ over the field $\mathbb{C}(v)$. Let $\mathbf{U}_{\mathbb{Z}[v,v^{-1}]}$ be the $\mathbb{Z}[v, v^{-1}]$-form in $\mathbf{U}_{\mathbb{C}(v)}$ constructed by Lusztig using divided powers.

- Set $\mathbf{U}_{\mathbb{Q}'_p} = \mathbf{U}_{\mathbb{Z}[v,v^{-1}]} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}'_p$ and $\mathbf{U}_{\mathbb{Q}'_p((z))}$ and

$\mathbf{U}_{\mathbb{A}((z))}$ etc. They all have compatible triangular decompositions.

- The subring $\mathbf{U}^0_{\mathbb{Z}[v,v^{-1}]}$ is a commutative and cocommutative Hopf algebra over $\mathbb{Z}[v,v^{-1}]$

- Each $\lambda = (\lambda_i) \in \mathbb{Q}'_p((z))^I$ defines a $\mathbb{Q}'_p((z))$- algebra homomorphism

$$\lambda : \mathbf{U}^0_{\mathbb{Q}'_p((z))} \to \mathbb{Q}'_p((z)) \quad K_i \mapsto v_i^{\lambda_i}.$$

Then $\lambda(\mathbf{U}_{\mathbb{A}[[z]]}) \subseteq \mathbf{A}[[z]]$ if $\lambda \in \mathbb{A}[[z]]^I$ and $\lambda(\mathbf{U}_{\mathbb{Z}_p[[z]]}) \subseteq \mathbb{Z}_p[[z]]$ if $\lambda \in \mathbb{Z}_p[[z]]^I$.

- For $\lambda \in \mathbb{Q}'((z))^I$, the quantum Verma module for the algebra $\mathbf{U}_{\mathbb{Q}_p'((z))}$ is

$$\mathbf{M}_{\mathbb{Q}_p'((z))}(\lambda) = \mathbf{U}_{\mathbb{Q}_p'((z))} \otimes_{\mathbf{U}^{\geq 0}_{\mathbb{Q}'((zz))}} \mathbb{Q}'((z))_\lambda$$

with irreducible quotient $\mathbf{L}_{\mathbb{Q}_p'((z))}(\lambda)$. The characters are similarly defined as functions $\mathbb{Q}_p'((z))^I \to \mathbb{Z}$.

- Standard argument implies $\mathbf{L}_{\mathbb{Q}_p'((z))}(\lambda) = \mathbf{M}_{\mathbb{Q}_p'((z))}(\lambda)$ unless $\langle \breve{\alpha}, \lambda + \rho \rangle \in \mathbb{Z}_{\geq 0} \subseteq \mathbb{Q}_p'((z))$. In general we have

$$\mathsf{ch}\, \mathbf{L}_{\mathbb{Q}_p'((z))}(\lambda) = \mathsf{ch}\, \Delta_{\mathbb{Q}_p'((z))}(\lambda).$$

Here $\Delta_{\mathbb{Q}_p'((z))}(\lambda)$ is the irreducible $\mathfrak{g}_{\mathbb{Q}_p'((z))}$-module.

- The characters $\mathsf{ch}\, \Delta_{\mathbb{Q}_p'((z))}(\lambda)$ can be determined by an argument similar that in the category $\mathcal{O}$ for $\mathfrak{g}_{\mathbb{C}}$ as outlined in Humphreys' book by replacing the field $\mathbb{C}$ with $\mathbb{Q}_p'((z))$.

• The generalized Kazhdan-Lusztig conjecture for non-regular blocks $(\mathcal{O}_{\mathbb{Q}_p'((z))})_\lambda$ gives the following decomposition of characters

$$\mathrm{ch}\,\mathbf{L}_{\mathbb{Q}_p'((z))}(\lambda) = \sum_\mu \mathbf{p}^0_{\mu,\lambda}\,\mathrm{ch}\,\mathbf{M}_{\mathbb{Q}_p'((z))}(\mu) \tag{1}$$

IV. Quantum groups at p^rth roots of unit over a p-adic field

- Let ξ be a p^rth root of 1.

- The map $\mathbb{Q}'_p[[z]] \to \mathbb{Q}'_p$ $(z \mapsto \xi - 1)$ induces $\mathbb{A}[[z]] \to \mathbb{A}$.

Define

$$\mathbf{U}_{\mathbb{Q}'_p} = \mathbf{U}_{\mathbb{Z}[v,v^{-1}]} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}'_p = \mathbf{U}_{\mathbb{A}[[z]]} \otimes_{\mathbb{A}[[z]]} \mathbb{Q}'_p$$

with $\mathbb{A}$-form $\mathbf{U}_{\mathbb{A}_p} = \mathbf{U}_{\mathbb{A}[[z]]} \otimes_{\mathbb{A}[[z]]} \mathbb{A}$ with tensor product decomposition

$$\mathbf{U}_{\mathbb{A}} = \mathbf{U}^{-}_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbf{U}^{0}_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbf{U}^{+}_{\mathbb{A}}.$$

- Let $\mathcal{O}_{\mathbb{Q}'_p}$ be the category $\mathcal{O}$ construction by Andersen and Mazorchuk for the quantum group $\mathbf{U}_{\mathbb{Q}'_p}$.

- The Verma module $\mathbf{M}_{\mathbb{Q}'_p}(\lambda)$ and irreducible quotient $\mathbf{L}_{\mathbb{Q}'_p}(\lambda)$ in $\mathcal{O}_{\mathbb{Q}'_p}$ with $\lambda \in X(\mathbb{Z}_p) \subseteq X(\mathbb{Q}'_p)$.

- For $\lambda \in X(\mathbb{Z}_p)$, $\mathbf{L}_{\mathbb{A}_p[[z]]}(\lambda) = \mathbf{U}_{\mathbb{A}_p[[z]]} v_\lambda^+ \subseteq \mathbf{L}_{\mathbb{Q}'_p((z))}(\lambda)$ is an $\mathbb{A}[[z]]$-lattice.

- Define $\mathbf{V}_{\mathbb{Q}'_p}(\lambda) = \mathbf{L}_{\mathbb{A}_p[[z]]}(\lambda) \otimes_{\mathbb{A}_p[[z]]} \mathbb{Q}'_p$ to be the Weyl module with the surjective maps $\mathbf{M}_{\mathbb{Q}'_p}(\lambda) \to \mathbf{V}_{\mathbb{Q}'_p}(\lambda) \to \mathbf{L}_{\mathbb{Q}'_p}(\lambda)$.

Proposition 1 (Andersen-Mazorchuk). *For any* $\lambda = \lambda' + p\lambda'' \in X(\mathbf{k})$ *with* $\lambda' \in X_1$,

$$L_{\mathbb{Q}'_p}(\lambda) = L_{\mathbb{Q}'_p}(\lambda') \otimes (\Delta_{\mathbb{Q}'_p}(\lambda''))^{(1)}.$$

Taking $\mathbb{A}$*-lattices generated by highest weight vectors and then tensor with* $\mathbb{A} \to \mathbf{k}$, *we get representations of* Dist$(\mathbf{G})$

Proposition 2. *For* $\lambda = \lambda' + p\lambda'' \in X(\mathbf{k})$,

$$\overline{L_{\mathbb{A}_p}(\lambda)} = \overline{L_{\mathbb{A}_p}(\lambda')} \otimes \Delta(\lambda'')^{(1)}.$$

V. Decomposition Multiplicities in Quantum Verma Modules

- For $\lambda \in X(\mathbf{Z}_p)$, define.

$$E^0_\lambda = \mathrm{ch}\,\Delta(\lambda) = \mathrm{ch}\,\Delta_{\mathbb{Q}'_p}(\lambda).$$

Here $\Delta_{\mathbb{Q}'_p}(\lambda)$ is the irreducible representation of the Lie algebra $\mathfrak{g}_{\mathbb{Q}'_p}$ with "$\mathbb{A}$-integral" highest weight λ.

- For each $r \geq 0$, any $\lambda \in X(\mathbf{Z}_p)$ can be uniquely written as $\lambda' + p^r\lambda''$ with $\lambda \in X_r$. Define recursively

$$E^{k+1}_\lambda = \sum_{\mu \in X(\mathbf{k})} \mathbf{p}_{\mu,\lambda''} E^k_{\lambda' + (p)^k\mu}. \tag{2}$$

Standard argument by Lusztig to get:

$$E^k_\lambda = \sum_{\mu \in X(\mathbf{k})} d^q_{\mu,\lambda''} E^{k+1}_{\lambda' + p^r \mu};$$

$$E^k_\lambda = E^1_{\lambda^0} (E^1_{\lambda^1})^{(1)} \cdots (E^1_{\lambda^{k-1}})^{(k-1)} (E^0_{\sum_{j \geq k} p^{r(j-k)} \lambda^j})^{(k)}.$$

- Define

$$E^\infty_\lambda = E^1_{\lambda^0} (E^1_{\lambda^1})^{(1)} \cdots (E^1_{\lambda^{k-1}})^{(k-1)} (E^1_{\lambda^k})^{(k)} \cdots . \qquad (3)$$

- Recursively define F^k_λ as follows: $F^0_\lambda = \operatorname{ch} M(\lambda)$ and for $k \geq 0$

$$F^{k+1}_\lambda = \sum_{\mu \in X(\mathbf{k})} a^q_{\mu,\lambda''} F^k_{\lambda' + (p)^k \mu}. \qquad (4)$$

Lusztig's argument implies

$$F^k_\lambda = \sum_{\mu \in X(\mathbf{k})} d^q_{\mu,\lambda''} F^{k+1}_{\lambda' + p^r \mu}.$$

$$F^k_\lambda = F^1_{\lambda^0} (F^1_{\lambda^1})^{(1)} \cdots (F^1_{\lambda^{k-1}})^{(k-1)} (F^0_{\sum_{j \geq k} p^{r(j-k)} \lambda^j})^{(k)}.$$

- As before, the infinite product converges in $F[X(\mathbf{k})]$. Note that $E^1_\lambda = F^1_\lambda = \mathrm{ch}\, L_q(\lambda)$ for all λ. We have $E^\infty_\lambda = F^\infty_\lambda$. But for other k, E^k_λ and F^k_λ are different.

Proposition 3. *For any k, both sets $\{E^k_\lambda \mid \lambda \in X(\mathbf{Z}_p)\}$ and $\{F^k_\lambda \mid \lambda \in X(\mathbf{Z}_p)\}$ are basis of $F[X(\mathbf{Z}_p)]$.*

We define that following decomposition of characters

$$F_\lambda^k = \sum_{\mu \in X(\mathbf{Z}_p)} d_{\mu,\lambda}^{(k)} E_\mu^k;$$
$$E_\lambda^k = \sum_{\mu \in X(\mathbf{Z}_p)} a_{\mu,\lambda}^{(k)} F_\mu^k$$

- For each fixed k and $\lambda \in X(\mathbf{k})$, define

$$\Delta^k(\lambda) = L(\lambda^0) \otimes \cdots \otimes L(\lambda^{k-1})^{(k-1)} \otimes (\Delta(\sum_{j \geq k} p^{j-k}\lambda^j))^{(k)}.$$

Then $\mathrm{ch}\, \Delta^k(\lambda) = E_\lambda^k$.

- $\Delta^{k+1}(\lambda)$ is a quotient of Δ_λ^k to get surjective maps of Dist($\mathbf{G}$)-modules

$$\Delta(\lambda) = \Delta^0(\lambda) \to \cdots \to \Delta^k(\lambda) \to \cdots \to \Delta^\infty(\lambda) = L(\lambda).$$

• For fixed k and $\lambda \in X(\mathbf{k})$, define a highest weight module

$$M_\lambda^k = L(\lambda^0) \otimes \cdots \otimes L(\lambda^{k-1})^{(k-1)} \otimes (M(\sum_{j \geq k} p^{j-k} \lambda^j))^{(k)}.$$

Then $\operatorname{ch} M_\lambda^k = F_\lambda^k$. Furthermore, we have surjective maps of Dist($\mathbf{G}$)-modules

$$M(\lambda) = M^0(\lambda) \to \cdots \to M^k(\lambda) \to \cdots \to M^\infty(\lambda) = L(\lambda).$$

THANK YOU!