The Jacobian, reflection arrangement and discriminant for reflection Hopf algebras
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Classical Invariant Theory:
Group G acting linearly on the algebra $\mathbb{k}[x_1, \ldots, x_n]$ and study $\mathbb{k}[x_1, \ldots, x_n]^G$.

Noncommutative Invariant Theory:
Replace:

$\mathbb{k}[x_1, \ldots, x_n]$ with appropriate noncommutative algebra A

G with a group (or Hopf algebra) that acts on A

to extend classical results.

Shephard-Todd-Chevalley Theorem

Let $\mathbb{k}$ be a field of characteristic zero.

Theorem (1954). The ring of invariants $\mathbb{k}[x_1, \cdots, x_n]^G$ under a finite group G is a polynomial ring if and only if G is generated by reflections.

A linear map g on V is called a reflection of V if all but one of the eigenvalues of g are 1,
i.e. dim $V^g =$ dim $V - 1$.

Example: Transposition permutation matrices are reflections, and S_n is generated by reflections.

Examples of reflection groups:

(1). $G = S_n$ on $\Bbbk[x_1 \dots, x_n]$.

(2). $G = D_{2n} = \{\rho, r : \rho^n = e = r^2, r\rho = \rho^{-1}r\}$ acts on $\Bbbk[x, y]$ as

$$\rho = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(3). $G = \langle g \rangle$ acts on $\Bbbk[x, y]$ as

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon_n \end{pmatrix}.$$

The Jacobian

When $A = \Bbbk[x_1, \ldots, x_n]$ and G is a reflection group then $A^G = \Bbbk[f_1, \ldots, f_n]$, and the Jacobian is

$$J := \det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^{n}.$$

Steinberg's Theorem (1960): J is the product (with multiplicities) of the reflecting hyperplanes.

Examples: A commutative

S_3 acts on $\Bbbk[x_1, x_2, x_3]$ then $f_1 = x_1 + x_2 + x_3$, $f_2 = x_1x_2 + x_1x_3 + x_2x_3$, $f_3 = x_1x_2x_3$ and

$$J = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

D_{2n} acts on $\Bbbk[x, y]$ then $f_1 = x^n + y^n$, $f_2 = xy$

$$J = n(x^n - y^n) =_{\Bbbk^\times} x^n - y^n$$
$$= (x - y)(x - \epsilon_n y) \dots (x - \epsilon_n^{n-1} y).$$

Let g act on $A = \Bbbk[x, y]$ by $g \cdot x = x$ and $g \cdot y = \epsilon_n y$. Then $A^{\langle g \rangle} = \Bbbk[x, y^n]$, and $J =_{\Bbbk^\times} y^{n-1}$.

$A = \Bbbk[x_1, \ldots, x_n]$ and G is a reflection group:

$\deg J = -n + \sum \deg(f_i)$ = number of reflections

$$g \cdot J = \det(g)^{-1} J$$

$$A_{\det^{-1}} = \{a \in A : g \cdot a = \det g^{-1} \cdot a\} = J A^G$$

Example: S_3 acts on $\Bbbk[x_1, x_2, x_3]$

$$J = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

$$\deg J = -3 + \sum \deg(f_i)$$

$-3 + 1 + 2 + 3$ = number of reflections

The Reflection Arrangement

Let a be the product of the distinct linear forms corresponding to the reflecting hyperplanes,

then $g \cdot \mathsf{a} = \det(g)\mathsf{a}$
and $A_{\det} = \mathsf{a}R$.

The Discriminant

$$\delta = \mathsf{a}J \in A^G$$

Example: S_3 acts on $\Bbbk[x_1, x_2, x_3]$

$$\mathsf{a} = J = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

$$\delta = J^2 = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 \in A^G.$$

$\Bbbk = \mathbb{C}$ Pairs (H, A) and action H on A.

- A is a Noetherian AS regular domain generated in degree 1
- H is a Hopf algebra acting on A:
  - H is semisimple Hopf algebra
  - H preserves the grading on A
  - A is an H-module algebra
  - The action of H on A is inner-faithful
- $A^H = \{a \in A \mid h.a = \epsilon(h)a$ for all $h \in H\}$.

We call H a reflection Hopf algebra for A if H acts on A so that A^H is AS regular.

Examples:

(a)$(\Bbbk G, \Bbbk[x_1, \ldots, x_n])$ G classical reflection group (Shephard-Todd-Chevalley Theorem).

(b)$(\Bbbk G, \Bbbk_{-1}[x_1, \ldots, x_n])$ G mystic reflection group

(c) $(\Bbbk^G, \Bbbk_{-1}[x, z][y; \sigma])$ with $G = D_8$

(d) $(H_8, \Bbbk_{-1}[x, y])$ and $(H_8, \Bbbk_{\pm i}[x, y])$.

Noncommutative Setting

Assume H is a reflection Hopf algebra for A,

$K = H^*$ and $R = A^H$.

For every $g \in G(K)$ define

$$A_g := \{a \in A \mid \rho(a) = a \otimes g\}$$

$$\mathrm{hdet} \in K$$
$$\mathrm{hdet} \in G(K).$$

Theorem: When $R := A^H$ AS regular,
there are elements in A unique up to scalars:

Jacobian $\mathsf{j}_{A,H} \in A$:
$A_{\mathrm{hdet}^{-1}}$ rank 1 free R-module generated by $\mathsf{j}_{A,H}$

Reflection Arrangement $\mathsf{a}_{A,H} \in A$:
A_{hdet} is rank 1 free R-module generated by $\mathsf{a}_{A,H}$

Discriminant $\delta_{A,H} \in R$:

$$\delta_{A,H} = \mathsf{j}_{A,H}\mathsf{a}_{A,H}$$

Candidates for <u>reflecting hyperplanes</u>:

$$\mathfrak{R}^l(f) := \{\mathbb{k}v \mid v \in A_1, v f_v = f \text{ for some } f_v \in A\}$$

$$\mathfrak{R}^r(f) := \{\mathbb{k}v \mid v \in A_1, f_v v = f \text{ for some } f_v \in A\}$$

- $\mathsf{a}_{A,H}$ divides $\mathsf{j}_{A,H}$ from the left and the right.
- When gldim A = 2 and H is either commutative or cocommutative, then
$$\mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^r(\mathsf{a}_{A,H}) = \mathfrak{R}^l(\mathsf{j}_{A,H}) = \mathfrak{R}^r(\mathsf{j}_{A,H}).$$
- For $H = \Bbbk^G$, both $\mathsf{j}_{A,H}$ and $\mathsf{a}_{A,H}$ are products of elements of degree one.

H semisimple $\Rightarrow$

$$H = \Bbbk \oplus \Bbbk \oplus \cdots \oplus \Bbbk \oplus M_{r_{n+1}}(\Bbbk) \oplus \cdots \oplus M_{r_{N-1}}(\Bbbk) \oplus M_{r_N}(\Bbbk)$$

where n summands are $\Bbbk$ and

$$H = p_1 H \oplus \cdots \oplus p_n H \oplus p_{n+1} H \oplus \cdots, \oplus p_N H$$

for central idempotents $p_i, i = 1 \ldots, N$.

$$I = M_{r_{n+1}}(\Bbbk) \oplus M_{r_{N-1}}(\Bbbk) \oplus M_{r_N}(\Bbbk)$$

$$H/I \cong \Bbbk G^*.$$

Let $K = H^*$, then $G \cong G(K)$ and $g \in G$ corresponds to a central idempotent $p_g = p_i$ for $i = 1, \ldots n$.

$$A = p_1 A \oplus \cdots \oplus p_n A \oplus p_{n+1} A \oplus \cdots, \oplus p_N A.$$

Then $A_g = p_g A$.

Each A_g is free of rank 1 over $R = A^H$.

$A_{\mathrm{hdet}^{-1}} = \mathsf{j}_{A,H} R$ and $A_{\mathrm{hdet}} = \mathsf{a}_{A,H} R$.

Example 1: $A = \Bbbk_{-1}[x, y]$ and $G = M(2, \alpha, \beta)$
Case $\alpha = 1$ (Binary dihedral groups):

$$\mathsf{a}_{A,H} = \mathsf{j}_{A,H} = (x^\beta - y^\beta).$$

deg $\mathsf{j}_{A,H}$ = # mystic reflections

Further,

$$\mathfrak{R}^l(\mathsf{j}_{A,H}) = \mathfrak{R}^r(\mathsf{j}_{A,H}) = \mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^r(\mathsf{a}_{A,H})$$

$$= \{\Bbbk(x + \xi y) \mid \xi^\beta = 1\}.$$

Group coaction case (H commutative)

Example 2: $A = \Bbbk_{-1}[x,z][y:\sigma]$, $D_8 = \langle r, \rho \rangle$, $zx = -xz, yx = zy, yz = xy$ and $H = \Bbbk^{D_8}$,

$$\deg_G(x) = r, \quad \deg_G(y) = r\rho, \quad \deg_G(z) = r\rho^2.$$

$A^H = \Bbbk[x^2, y^2, z^2]$. $\quad \mathrm{hdet} = \mathrm{hdet}^{-1} = r\rho^3$

$$\mathsf{j}_{A,H} = \mathsf{a}_{A,H} =_{\Bbbk^\times} zxy =_{\Bbbk^\times} zyz =_{\Bbbk^\times} xyx =_{\Bbbk^\times} xzy$$

$$=_{\Bbbk^\times} yzx =_{\Bbbk^\times} yxz, \quad \delta_{A,H} = x^2y^2z^2$$

$$\mathfrak{R}^l(\mathsf{j}_{A,H}) = \mathfrak{R}^l(\mathsf{a}_{A,H}) = \mathfrak{R}^r(\mathsf{j}_{A,H}) = \mathfrak{R}^r(\mathsf{a}_{A,H})$$

$$= \{\Bbbk x, \Bbbk y, \Bbbk z\}$$

Example 3: H_8 representation on $V = \Bbbk u \oplus \Bbbk v$:

$$x \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

determines the H_8-action on $A = \Bbbk_i[u, v]$.

$G(K) = \{1, g, g', gg'\}$ the Klein-4 group,

$$\text{hdet}^{-1} = \text{hdet} = gg'$$

$$p_1 A = A^{H_8} = R = \Bbbk[u^2+v^2, u^2v^2]$$
$$p_g A = (u^2 - v^2)R$$
$$p_{g'} A = (uv)R$$
$$p_{gg'} A = (u^3v + uv^3)R = (uv(u^2 - v^2))R.$$

$$\mathsf{a}_{A,H_8} = \mathsf{j}_{A,H_8} =_{\Bbbk^\times} uv(u^2 - v^2)$$
$$\delta_{A,H_8} =_{\Bbbk^\times} u^2v^2(u^2 - v^2)^2$$
$$= u^2v^2[(u^2+v^2)^2 - 4u^2v^2] \in R$$

$$\mathfrak{R}^l(\mathsf{a}_{A,H_8}) = \mathfrak{R}^l(\mathsf{j}_{A,H_8})$$
$$= \{\mathbb{k}u, \mathbb{k}v, \mathbb{k}(u + e^{\frac{3}{8}(2\pi i)}v), \mathbb{k}(u + e^{\frac{7}{8}(2\pi i)}v)\}$$

and

$$\mathfrak{R}^r(\mathsf{a}_{A,H_8}) = \mathfrak{R}^r(\mathsf{j}_{A,H_8})$$
$$= \{\mathbb{k}u, \mathbb{k}v, \mathbb{k}(u + e^{\frac{1}{8}(2\pi i)}v), \mathbb{k}(u + e^{\frac{5}{8}(2\pi i)}v)\}.$$

Questions:

- $\mathfrak{R}^l(\mathsf{a}_{AH}) = \mathfrak{R}^l(\mathsf{j}_{A,H}) = \mathfrak{R}^l(\delta_{A,H})$?
- Is $\mathfrak{R}^l(\mathsf{j}_{A,H})$ isomorphic to $\mathfrak{R}^r(\mathsf{j}_{A,H})$?
- Is there a noncommutative notion of hyperplane arrangement?
- Classical case:
  deg a = number of reflecting hyperplanes and deg j = number of reflections in G.
  Meaning of degrees in noncommutative case?

- Role of $\mathsf{a}_{A,H}$, $\mathsf{j}_{A,H}$, and $\delta_{A,H}$ in representation theory of H and classification (up to dual cycle twists) of reflection Hopf algebra pairs (H, A) for A of dimension 2.

THANKS!