The Picture Space of a Gentle Algebra

The Story of a Counterexample

Eric J Hanson

Brandeis University

Joint work with Kiyoshi Igusa

Maurice Auslander Distinguished Lectures and International Conference
April 29, 2019

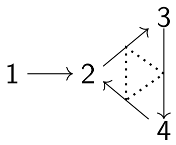
Outline

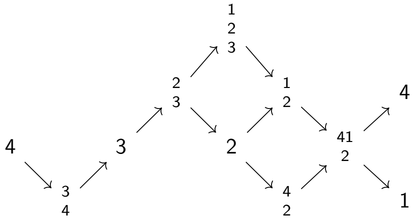
The Setup

- Let Λ be a finite dimensional, basic algebra over an arbitrary field K.
- Denote by modΛ the category of finitely generated (right) Λ-modules.
- All subcategories are assumed full and closed under isomorphisms.
- τ is the Auslander-Reiten translate and $(-)[1]$ is the shift functor.
- $S \in$ modΛ (or $\mathcal{D}^b($mod$\Lambda))$ is called a *brick* if $\mathrm{End}(S)$ is a division algebra. A collection of Hom-orthogonal bricks is called a *semibrick*.

Our Main Example

Let A be the K-algebra whose (bounded) quiver and AR quiver are:





Properties of A:

- It is representation (hence τ-tilting) finite.
- Every indecomposable A-module is a brick.
- It is cluster tilted of type A_4.
- It is a gentle algebra.
- It is not hereditary.

Picture Groups

Recall a subcategory $\mathcal{T} \subset \mathrm{mod}\Lambda$ is a *torsion class* if it is closed under extensions and quotients. We assume $\mathrm{mod}\Lambda$ contains only finitely many torsion classes (i.e., Λ is τ-tilting finite DIJ '15).

Theorem (Barnard-Carroll-Zhu '17[1])

Suppose $\mathcal{T}' \subsetneq \mathcal{T}$ is a minimal inclusion of torsion classes. Then there exists a unique brick $S \in \mathcal{T} \setminus \mathcal{T}'$ such that $\mathcal{T} = \mathrm{Filt}(\mathcal{T}' \cup \{S\})$, called the ***brick label*** *of the inclusion.*

[1] This brick labeling is also constructed by Asai, Brüstle-Smith-Treffinger, and Demonet-Iyama-Reading-Reiten-Thomas.

Picture Groups

The definition of the picture group was first given by Igusa-Todorov-Weyman '16.

Definition

The *picture group* of Λ, denoted $G(\Lambda)$, is the finitely presented group with the following presentation.

- For every brick $S \in \mathrm{mod}\Lambda$, there is a generator X_S.
- For every torsion class $\mathcal{T}$, there is a generator $g_{\mathcal{T}}$.
- There is a relation $g_0 = e$.
- For every minimal inclusion of torsion classes $\mathcal{T}' \subsetneq \mathcal{T}$ labeled by S, there is a relation $g_{\mathcal{T}} = X_S g_{\mathcal{T}'}$.

The Picture Space

- Igusa, Todorov and Weyman also associate to Λ a topological space called the *picture space* of Λ.
- This space can be defined as the classifying space of the τ-cluster morphism category of Λ, defined by Buan and Marsh in '18 to generalize a construction of Igusa and Todorov in '17.

Theorem (H-Igusa '18)

Let Λ be an arbitrary τ-tilting finite algebra. Then

1. *The fundamental group of the picture space is $G(\Lambda)$.*
2. *If the 2-simple minded collections for Λ can be defined using pairwise compatibility conditions (plus one technical condition) then the picture space is a $K(G(\Lambda), 1)$.*

2-Simple Minded Collections and Semibrick Pairs

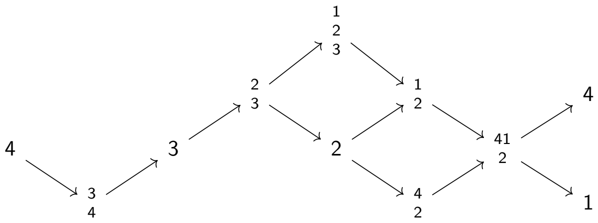
Definition-Theorem (Brüstle-Yang '13)

Let $\mathcal{X} = \mathcal{S}_p \sqcup \mathcal{S}_n[1]$ with $\mathcal{S}_p, \mathcal{S}_n$ semibricks in modΛ. Then $\mathcal{X}$ is called a *2-simple minded collection* if

1. For all $S \in \mathcal{S}_p, T \in \mathcal{S}_n, \mathrm{Hom}(S, T) = 0 = \mathrm{Ext}(S, T)$.
2. The smallest subcategory of $\mathcal{D}^b$(modΛ) containing $\mathcal{X}$ and closed under triangles, direct summands, and shifts is $\mathcal{D}^b$(modΛ).

- If only (1) holds, we will call $\mathcal{X}$ a *semibrick pair*. We call a semibrick pair *completable* if it is contained in a 2-simple minded collection.
- Being a semibrick pair is a pairwise condition! Being completable...?

2-Simple Minded Collections and Semibrick Pairs



Example

- $S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4$ and $S_1[1] \sqcup S_2[1] \sqcup S_3[1] \sqcup S_4[1]$ are 2-simple minded collections.
- $S_1 \sqcup S_3 \sqcup {}^{1}_{2}[1] \sqcup {}^{3}_{4}[1]$ is a 2-simple minded collection.
- ${}^{4}_{2} \sqcup {}^{2}_{3}[1]$ is a semibrick pair which is not completable.

(nontrivial) Fact: if Λ is hereditary, then every semibrick pair is completable.

The Pairwise 2-Simple Minded Compatibility Property

Definition

The algebra Λ does **NOT** have the *pairwise 2-simple minded compatibility property* if there exists a semibrick pair $\mathcal{X}$ which is not completable such that for every pair $S, T \in \mathcal{X}$, the semibrick pair $S \sqcup T$ is completable.

Approximations

Proposition (Brüstle-Yang '13)

Let $\mathcal{X} = \mathcal{S}_p \sqcup \mathcal{S}_n[1]$ be a completable semibrick pair. Let $S \in \mathcal{S}_p$ and $T \in \mathcal{S}_n$. Then every left minimal $(\mathsf{Filt} S)$*-approximation $g^+_{ST} : T \to S_T$ is either mono or epi.*

If $\dim\mathrm{Hom}(T, S) = 1$ then g^+_{ST} is just a morphism $T \to S$.

Natural question: Is every semibrick pair with this property completable? No.

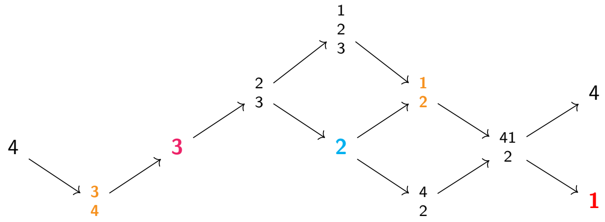
Mutation

Definition

Let $\mathcal{X} = \mathcal{S}_p \sqcup \mathcal{S}_n[1]$ be a 2-simple minded collection and let $S \in \mathcal{S}_p$. The *left mutation* of $\mathcal{X}$ at S, denoted $\mu_S^+(\mathcal{X})$, is the new collection defined as follows.

- $\mu_S^+(S) = S[1]$.
- For all other $T \in \mathcal{X}$, $\mu_S^+(T) = \text{cone}(g_{ST}^+)$, where $g_{ST}^+ : T[-1] \to S_T$ is a minimal left $(\text{Filt}S)$-approximation.

Mutation



Example

$\mu^+_{S_1}\left(S_1 \sqcup S_3 \sqcup {}^1_2[1] \sqcup {}^3_4[1]\right) = S_3 \sqcup S_1[1] \sqcup S_2[1] \sqcup {}^3_4[1]$:

- $\mathrm{Hom}(S_3[-1], S_1) = \mathrm{Ext}(S_3, S_1) = 0$.
- $\mathrm{Hom}\left({}^3_4, S_1\right) = 0$.
- The nonzero morphism ${}^1_2 \to S_1$ is epi with kernel S_2.

Mutation

Definition

Let $\mathcal{X} = \mathcal{S}_p \sqcup \mathcal{S}_n[1]$ be a 2-simple minded collection and let $S \in \mathcal{S}_p$. The *left mutation* of $\mathcal{X}$ at S, denoted $\mu_S^+(\mathcal{X})$, is the new collection defined as follows.

- $\mu_S^+(S) = S[1]$.
- For all other $T \in \mathcal{X}$, $\mu_S^+(T) = \text{cone}(g_{ST}^+)$, where $g_{ST}^+ : T[-1] \to S_T$ is a minimal left $(\text{Filt}S)$-approximation.

Observation: This is a 'pairwise' definition: $\mu_S^+(T)$ depends only on S and T.

Definition

Let $\mathcal{X} = \mathcal{S}_p \sqcup \mathcal{S}_n[1]$ be a semibrick pair. Then we can define the *left mutation* of $\mathcal{X}$ at $S \in \mathcal{S}_p$ using the above formula.

Mutation

Proposition (H-Igusa '19)

Let $\mathcal{X} = \mathcal{S}_p \sqcup \mathcal{S}_n[1]$ be a semibrick pair and let $S \in \mathcal{S}_p$.

1. *For all $T \in \mathcal{X}$, the object $\mu_S^+(T)$ is a brick.*
2. *Assume for all $T \in \mathcal{S}_n$ the minimal left $(\mathrm{Filt}S)$-approximation g_{ST}^+ is either mono or epi. Then $\mu_S^+(\mathcal{X})$ is a semibrick pair.*
3. *$\mathcal{X}$ is completable if and only if $\mu_S^+(\mathcal{X})$ is completable.*

Natural question: Assume (2) and let $S' \in \mu_S^+(\mathcal{X})_p$. Is $\mu_{S'}^+ \circ \mu_S^+(\mathcal{X})$ always a semibrick pair? No.

Determining Completability

Theorem (Asai '16)

Let $\mathcal{X} = \mathcal{S}_p \sqcup \mathcal{S}_n[1]$ be a semibrick pair. If $\mathcal{S}_p = \emptyset$ or $\mathcal{S}_n = \emptyset$, (i.e. either $\mathcal{X} = \mathcal{S}_n[1]$ or $\mathcal{X} = \mathcal{S}_p$) then $\mathcal{X}$ is completable.

Strategy: Start with an arbitrary semibrick pair $\mathcal{X}$. If we mutate enough times, one of the following things will happen:

1. We will reach a semibrick pair $\mathcal{Y} = \mathcal{S}_n[1]$, which we know is completable.
2. We will reach a semibrick pair $\mathcal{Y} = \mathcal{S}_p \sqcup \mathcal{S}_n[1]$ containing some $S \in \mathcal{S}_p$ and $T \in \mathcal{S}_n$ for which the minimal left $(\text{Filt} S)$-approximation g^+_{ST} is neither mono nor epi, which we know is not competable.

The Hereditary Case

Theorem (Igusa-Todorov '17)

Suppose Λ is (representation finite) hereditary. Then Λ has the 2-simple minded pairwise compatibility property.

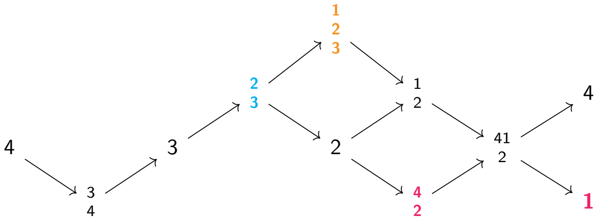
Key Observation for the New Proof.

Let $f : M \to N$ be any morphism. Then

$$\text{cone}(f) = \ker(f)[1] \sqcup \operatorname{coker}(f).$$

In particular, cone(f) can only be a brick if f is either mono or epi. Thus, given a semibrick pair $\mathcal{X} = \mathcal{S}_p \sqcup \mathcal{S}_n[1]$ and any $S \in \mathcal{S}_p$ and $T \in \mathcal{S}_n[1]$, the minimal left (Filt$S$)-approximation g^{+}_{ST} is either mono or epi. □

The Counterexample

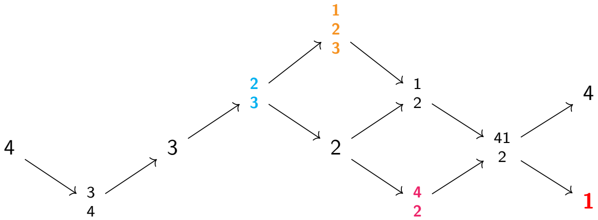


Consider the semibrick pair $\mathcal{X} = 1 \sqcup \begin{smallmatrix}4\\2\end{smallmatrix} \sqcup \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1]$.

Each pair of $\mathcal{X}$ is completable:

1. $1 \sqcup \begin{smallmatrix}4\\2\end{smallmatrix}$ has $\mathcal{S}_n = \emptyset$.
2. Mutate $1 \sqcup \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1]$ at 1 to obtain $1[1] \sqcup \begin{smallmatrix}2\\3\end{smallmatrix}[1]$. This has $\mathcal{S}_p = \emptyset$.
3. Mutate $\begin{smallmatrix}4\\2\end{smallmatrix} \sqcup \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1]$ at $\begin{smallmatrix}4\\2\end{smallmatrix}$ to obtain $\begin{smallmatrix}4\\2\end{smallmatrix}[1] \sqcup \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1]$. This has $\mathcal{S}_p = \emptyset$.

The Counterexample



Consider the semibrick pair $\mathcal{X} = 1 \sqcup \begin{smallmatrix}4\\2\end{smallmatrix} \sqcup \begin{smallmatrix}1\\2\\3\end{smallmatrix}[1]$.

- Mutate at 1 to obtain $\begin{smallmatrix}4\\2\end{smallmatrix} \sqcup 1[1] \sqcup \begin{smallmatrix}2\\3\end{smallmatrix}[1]$.
- The map $\begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}4\\2\end{smallmatrix}$ is neither mono nor epi, so $\mathcal{X}$ is not contained in a 2-simple minded collection!

The General Result

Theorem (H-Igusa '19)

Let $\Lambda = KQ/I$ be a τ-tilting finite gentle algebra such that Q contains no loops or 2-cycles. Then Λ has the pairwise 2-simple minded compatibility property if and only if every vertex of Q has degree at most 2.

Corollary

If Λ is cluster tilted of type A_n and not hereditary, then Λ has the 2-simple minded compatibility property if and only if $n = 3$.

The General Result

It is less clear what happens if we allow loops or 2-cycles. For example:

Proposition (Barnard-H, April 28 2019)

Consider the algebra $\Lambda = KQ/I$ where Q is the quiver

$$1 \leftrightarrows 2 \leftrightarrows \cdots \leftrightarrows n$$

and I is generated by all 2-cycles. Then Λ has the pairwise 2-simple minded compatibility property if and only if $n \leq 3$.

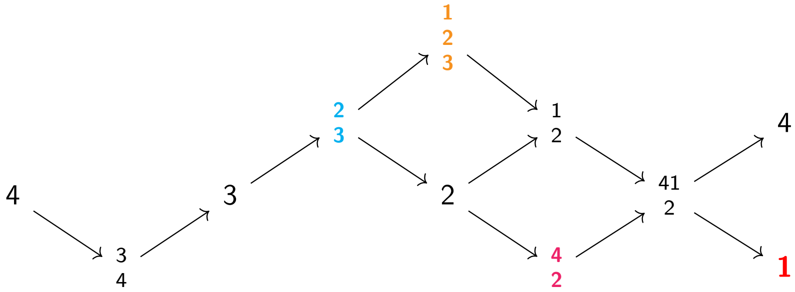
Corollary

The preprojective algebra of type A_n has the pairwise 2-simple minded compatibility property if and only if $n \leq 3$.

Thank You!

arXiv:1809.08989 - τ-Cluster Morphism Categories and Picture Groups

arXiv:1904.03166 - Pairwise Compatibility for the 2-Simple Minded Collections of Gentle Algebras



References I

Sota Asai.
Semibricks.
arXiv, 1610.05860, 2017.

Emily Barnard, Andrew T. Carroll, and Shijie Zhu.
Minimal inclusions of torsion classes.
arXiv, 1710.08837, 2017.

Thomas Brüstle, David Smith, and Hipolito Treffinger.
Wall and chamber structure for finite-dimensional algebras.
arXiv, 1805.01880, 2018.

Thomas Brüstle and Dong Yang.
Ordered exchange graphs.
arXiv, 1302.6045, 2015.

Aslak Bakke Buan and Robert J. Marsh.
A category of wide subcategories.
arXiv, 1802.03812, 2018.

Laurent Demonet, Osamu Iyama, and Gustavus Jasso.
τ-tilting finite algebras, bricks, and g-vectors.
Int. Math. Res. Notices, rnx135, 2017.

References II

Laurent Demonet, Osamu Iyama, Nathan Reading, Idun Reiten, and Hugh Thomas.
Lattice theory of torsion classes.
arXiv, 1711.01785, 2018.

Eric J. Hanson and Kiyoshi Igusa.
τ-cluster morphism categories and picture groups.
arXiv, 1809.08989, 2018.

Eric J. Hanson and Kiyoshi Igusa.
Pairwise compatibility for the 2-simple minded collections of gentle algebras.
arXiv, 1904.03166, 2019.

Kiyoshi Igusa and Gordana Todorov.
Signed exceptional sequences and the cluster morphism category.
arXiv, 1706.02041, 2017.

Kiyoshi Igusa, Gordana Todorov, and Jerzy Weyman.
Picture groups of finite type and cohomology in type A_n.
arXiv, 1609.02636, 2016.