Shapes of the irreducible morphisms and Auslander-Reiten Triangles in the stable category of modules over repetitive algebras

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Road map

Section

Repetitive Algebras

- Let A be a finite-dimensional k-algebra over field k.
- For simplicity, we assume that A is basic and k is algebraically closed.
- Denote by $D = \mathrm{Hom}_k(-, k)$ the standard duality on A-mod.

Let us construct the **repetitive algebra** $\widehat{A}$ of A as proposed by D. Hughes and J. Waschbüsch (1983).

- The underlying vector space of repetitive algebra $\widehat{A}$ is given by

$$\widehat{A} = (\oplus_{i\in\mathbb{Z}} A) \oplus (\oplus_{i\in\mathbb{Z}} DA),$$

  $\widehat{a} = (a_i, \varphi_i)_{i\in\mathbb{Z}}$ with $a_i \in A$, $\varphi_i \in DA$ and almost all a_i, φ_i being zero.
- The multiplication is defined by

$$\widehat{a} \cdot \widehat{b} = (a_i, \varphi_i)_{i\in\mathbb{Z}} \cdot (b_i, \psi_i)_{i\in\mathbb{Z}} = (a_i b_i, a_{i+1}\psi_i + \varphi_i b_i)_{i\in\mathbb{Z}}.$$

A $\widehat{A}$-module $M = (M_i, f_i)_{i\in\mathbb{Z}}$, where the M_i are A-modules, all but finitely many being zero (finitely generated left module), the f_i are A-homomorphims $f_i : DA \otimes_A M_i \longrightarrow M_{i+1}$, such that $f_{i+1}(1 \otimes f_i) = 0$ for all $i \in \mathbb{Z}$.

Instead of $M = (M_i, f_i)_{i\in\mathbb{Z}}$ we also write:

$$M : \qquad \cdots \rightsquigarrow M_{i-1} \overset{f_{i-1}}{\rightsquigarrow} M_i \overset{f_i}{\rightsquigarrow} M_{i+1} \rightsquigarrow \cdots .$$

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A $\widehat{A}$-homomorphism $h : M = (M_i, f_i)_{i\in\mathbb{Z}} \longrightarrow N = (N_i, g_i)_{i\in\mathbb{Z}}$ between $\widehat{A}$-modules is a sequence $h = (h_i)_{i\in\mathbb{Z}}$ of A-homomorphims

$$\begin{array}{ccc} DA\otimes_A M_i & \xrightarrow{f_i} & M_{i+1} \\ \downarrow{\scriptstyle 1\otimes h_i} & & \downarrow{\scriptstyle h_{i+1}} \\ DA\otimes_A N_i & \xrightarrow{g_i} & N_{i+1}. \end{array}$$

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$$\begin{array}{ccccccccc} M: & \cdots \rightsquigarrow & M_{i-1} & \overset{f_{i-1}}{\rightsquigarrow} & M_i & \overset{f_i}{\rightsquigarrow} & M_{i+1} & \rightsquigarrow \cdots \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_{i-1}} & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle h_{i+1}} & \\ N: & \cdots \rightsquigarrow & N_{i-1} & \overset{g_{i-1}}{\rightsquigarrow} & N_i & \overset{g_i}{\rightsquigarrow} & N_{i+1} & \rightsquigarrow \cdots . \end{array}$$

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- We denoted by $\widehat{A}$-mod the category of finitely generated left modules over the repetitive algebra A.
- We denoted by $\widehat{A}$-$\underline{\text{mod}}$ the stable category of $\widehat{A}$-mod.

Section

Definition

An $\widehat{A}$-homomorphism $h = (h_i)_{i\in\mathbb{Z}} : M = (M_i, f_i)_{i\in\mathbb{Z}} \longrightarrow N = (N_i, g_i)_{i\in\mathbb{Z}}$:

1. *is called **smonic** (resp. **sepic**) if all its components h_i are split monomorphisms (resp. split epimorphisms) and*
2. *is called **sirreducible** if there is exactly one index ι_0 such that h_{ι_0} is irreducible morphism and h_i is a split epimorphism for $i < \iota_0$ and a split monomorphism for $i > \iota_0$.*

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Shapes of the irreducible morphisms

Theorem (-, 2017, [2])

Let $h : M = (M_i, f_i)_{i\in\mathbb{Z}} \longrightarrow N = (N_i, g_i)_{i\in\mathbb{Z}}$ *be an irreducible homomorphism in* $\widehat{A}$*-mod. Then one of the following conditions holds:*

1. h *is a smonic morphism;*
2. h *is a sepic morphism;*
3. h *is a sirreducible morphism.*

Irreducible smonic

$$\begin{array}{ccccccccccc}
M_{a-1} & \overset{f_{a-1}}{\rightsquigarrow} & M_a & \overset{f_a}{\rightsquigarrow} & M_{a+1} & \rightsquigarrow & \cdots & \rightsquigarrow & M_b & \overset{f_b}{\rightsquigarrow} & M_{b+1} \cdots \\
\downarrow 1 & & \downarrow (1,0)^t & & \downarrow (1,0)^t & & & & \downarrow (1,0)^t & & \downarrow (1,0)^t \\
M_{a-1} & \underset{d_{a-1}}{\rightsquigarrow} & M_a \oplus N'_a & \underset{d_a}{\rightsquigarrow} & M_{a+1} \oplus N'_{a+1} & \rightsquigarrow & \cdots & \rightsquigarrow & M_b \oplus N'_b & \underset{d_b}{\rightsquigarrow} & M_{b+1} \oplus N'_{b+1},
\end{array}$$

where $h_{[a,b]}$ is the mono heart of h.

For all $i < a-1$ we have that $d_i = f_i$ and $d_{a-1} = (f_{a-1}, 0)^t$.
For $a \leq i < b$,

$$d_i = \begin{pmatrix} f_i & b_i \\ 0 & \overline{g}_i \end{pmatrix}, \text{ with } b_i \neq 0 \text{ for all } a \leq i < b.$$

For all $i \geq b$,

$$d_i = \begin{pmatrix} f_i & 0 \\ 0 & \overline{g}_i \end{pmatrix}.$$

Irreducible sepic

$$
\begin{array}{ccccccccccc}
N_{a-1} \oplus M'_{a-1} & \overset{d_{a-1}}{\rightsquigarrow} & N_a \oplus M'_a & \overset{d_a}{\rightsquigarrow} & N_{a+1} \oplus M'_{a+1} & \rightsquigarrow & \cdots & \rightsquigarrow & N_b \oplus M'_b & \overset{d_b}{\rightsquigarrow} & N_{b+1} \\
\downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle (1,0)} & & & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle 1} \\
\cdots N_{a-1} & \underset{g_{a-1}}{\rightsquigarrow} & N_a & \underset{g_a}{\rightsquigarrow} & N_{a+1} & \rightsquigarrow & \cdots & \rightsquigarrow & N_b & \underset{g_b}{\rightsquigarrow} & N_{b+1},
\end{array}
$$

where $h_{[a,b]}$ is the epi heart of h.

For all $i > b$, we have that $d_i = g_i$ and $d_b = (g_b, 0)$.
For $a \leq i < b$,

$$d_i = \begin{pmatrix} g_i & 0 \\ c_i & f'_i \end{pmatrix}, \text{with } c_i \neq 0 \text{ for all } a \leq i < b.$$

For all $i < a$,

$$d_i = \begin{pmatrix} g_i & 0 \\ 0 & f'_i \end{pmatrix}.$$

Irreducible sirreducible (monomorphism)

$$
\begin{array}{ccccccccccccc}
\cdots & \rightsquigarrow & N_{k-2} & \overset{g_{k-2}}{\rightsquigarrow} & N_{k-1} & \overset{d_{k-1}}{\rightsquigarrow} & M_k & \overset{f_k}{\rightsquigarrow} & M_{k+1} & \overset{f_{k+1}}{\rightsquigarrow} & M_{k+2} & \rightsquigarrow & \cdots \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle (1,0)^t} & & \downarrow{\scriptstyle (1,0)^t} & & \\
\cdots & \rightsquigarrow & N_{k-2} & \underset{g_{k-2}}{\rightsquigarrow} & N_{k-1} & \underset{g_{k-2}}{\rightsquigarrow} & N_k & \underset{d_k}{\rightsquigarrow} & M_{k+1} \oplus N'_{k+1} & \underset{d_{k+1}}{\rightsquigarrow} & M_{k+2} \oplus N'_{k+2} & \rightsquigarrow & \cdots
\end{array}
$$

where h_k is an irreducible A-monomorphism.

For $i > k$,

$$d_i = \begin{pmatrix} f_i & 0 \\ 0 & \overline{g}_i \end{pmatrix}.$$

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Irreducible sirreducible (epimorphism)

$$\begin{array}{ccccccccccc}
\cdots & \rightsquigarrow & N_{k-2} \oplus M'_{k-2} & \overset{d_{k-2}}{\rightsquigarrow} & N_{k-1} \oplus M'_{k-1} & \overset{d_{k-1}}{\rightsquigarrow} & M_k & \overset{f_k}{\rightsquigarrow} & M_{k+1} & \overset{f_{k+1}}{\rightsquigarrow} & M_{k+2} \rightsquigarrow \\
 & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
\cdots & \rightsquigarrow & N_{k-2} & \underset{g_{k-2}}{\rightsquigarrow} & N_{k-1} & \underset{g_{k-2}}{\rightsquigarrow} & N_k & \underset{d_k}{\rightsquigarrow} & M_{k+1} & \underset{f_{k+1}}{\rightsquigarrow} & M_{k+2} \rightsquigarrow
\end{array}$$

where h_k is an irreducible A-epimorphism.

For $i < k$,

$$d_i = \begin{pmatrix} g_i & 0 \\ 0 & f'_i \end{pmatrix}.$$

Proposition

Let $h : M = (M_i, f_i)_{i\in\mathbb{Z}} \longrightarrow N = (N_i, g_i)_{i\in\mathbb{Z}}$ *be a homomorphism in* $\widehat{A}$*-mod, such that* M *and* N *have not projective summands and let* $\underline{h}$ *be its stable class in* $\widehat{A}$*-*$\underline{\text{mod}}$*. Then,* h *is split mono (resp. split epi) if and only if* $\underline{h}$ *is split mono (resp. split epi).*

Proposition

Let $h : M = (M_i, f_i)_{i\in\mathbb{Z}} \longrightarrow N = (N_i, g_i)_{i\in\mathbb{Z}}$ *be a homomorphism in* $\widehat{A}$*-mod, such that* M *and* N *have not projective summands and let* $\underline{h}$ *be its stable class in* $\widehat{A}$*-*$\underline{\text{mod}}$*. Then,* h *is irreducible if and only if* $\underline{h}$ *is irreducible.*

Section

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Theorem

1. *The category* $\widehat{A}$-mod *has almost split sequences (1983 D. Hughes and J. Waschbüsch).*
2. *The category* $\widehat{A}$-mod *is a Frobenius, and the category* $\widehat{A} - \underline{\text{mod}}$ *is triangulated (1988 D. Happel).*

Shapes of Auslander-Reiten Triangles

Theorem (-, Y. Calderón-Henao, and JA. Vélez-Marulanda, preprint, [1])

Let $M = (M_i, f_i)_{i\in\mathbb{Z}} \xrightarrow{u} N = (N_i, g_i)_{i\in\mathbb{Z}} \xrightarrow{v} L = (L_i, l_i)_{i\in\mathbb{Z}} \xrightarrow{w} T(M) \quad (1)$ *be an Auslander-Reiten triangle in* $\widehat{A} - \underline{mod}$*. Then there exist an almost split sequence*

$$0 \longrightarrow M \xrightarrow{\begin{bmatrix} u \\ i \end{bmatrix}} N \oplus P \xrightarrow{\begin{bmatrix} v \\ p \end{bmatrix}} L \longrightarrow 0$$

in $\widehat{A} - mod$*, with* P *an* $\widehat{A}-$ *projective module, such that the triangle induce by this sequence is isomorphic to* (1)*. If* $P \neq 0$*, then* P *is indecomposable,* $rad(P) \cong M$*, and* $L \cong P/soc(P)$*.*

Shapes of Auslander-Reiten Triangles

Theorem (-, Y. Calderón-Henao, and JA. Vélez-Marulanda, preprint, [1])

Let $M = (M_i, f_i)_{i\in\mathbb{Z}} \xrightarrow{u} N = (N_i, g_i)_{i\in\mathbb{Z}} \xrightarrow{v} L = (L_i, l_i)_{i\in\mathbb{Z}} \xrightarrow{w} T(M)$ *be an Auslander-Reiten triangle in* $\widehat{A} - \underline{mod}$*. Then*

1. *If* $\underline{u}$ *is smonic, then* $\underline{v}$ *is sepic.*
2. *If* $\underline{u}$ *is sepic, then* $\underline{v}$ *is sirreducible.*
3. *If* $\underline{u}$ *is sirreducible, then* $\underline{v}$ *is smonic or sirreducible.*

The quiver of a repetitive algebra

Theorem (1999 J. Schröer)

Let Q by a finite quiver, and let ρ be a set of relations for Q which are either zero-relations or commutativity-relations such that (Q, ρ) is locally bounded. Let $(\widehat{Q}, \widehat{\rho})$ be constructed as in (1999 J. Schröer). Then $k\widehat{Q}/ < \widehat{\rho} >$ is the repetitive algebra of $kQ/ < \rho >$.

Theorem (1991 C. M. Ringel and 1999 J. Schröer)

Let A be a finite-dimensional k-algebra. Then

$$A \text{ is gentle if and only if } \widehat{A} \text{ is special biserial.}$$

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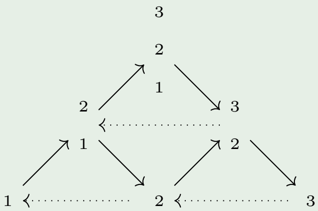
Example

Let A_1 be the finite dimensional algebra given by the quiver

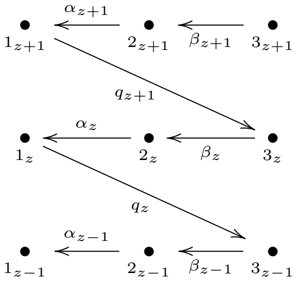
$Q: \underset{1}{\bullet} \xleftarrow{\alpha} \underset{2}{\bullet} \xleftarrow{\beta} \underset{3}{\bullet}$. The radical series of the indecomposables projective, injective and simples left A_1-modules are given as follows:

$$P_1 = 1, \quad S_2 = 2, \quad S_3 = 3, \quad P_2 = \begin{array}{c} 2 \\ 1 \end{array}, \quad P_3 = \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \; y \; I_2 = \begin{array}{c} 3 \\ 2 \end{array}$$

The Auslander-Reiten quiver of A_1 is the given as follows:

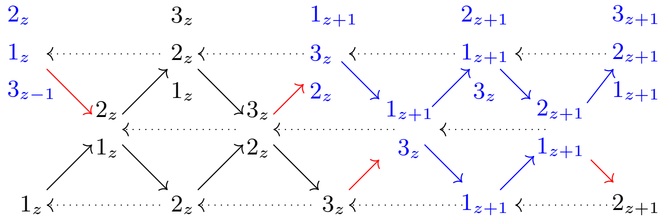


Recall that $\widehat{Q}$ is given by



$$P_{1_z} = \begin{matrix} 1_z \\ \downarrow \\ 3_{z-1} \\ \downarrow \\ 2_{z-1} \\ \downarrow \\ 1_{z-1} \end{matrix} = I_{1_{z-1}}, \quad P_{2_z} = \begin{matrix} 2_z \\ \downarrow \\ 1_z \\ \downarrow \\ 3_{z-1} \\ \downarrow \\ 2_{z-1} \end{matrix} = I_{2_{z-1}} \text{ and } \quad P_{3_z} = \begin{matrix} 3_z \\ \downarrow \\ 2_z \\ \downarrow \\ 1_z \\ \downarrow \\ 3_{z-1} \end{matrix} = I_{3_{z-1}}$$

The stable Auslander-Reiten quiver of $\widehat{A_1}$ is given by



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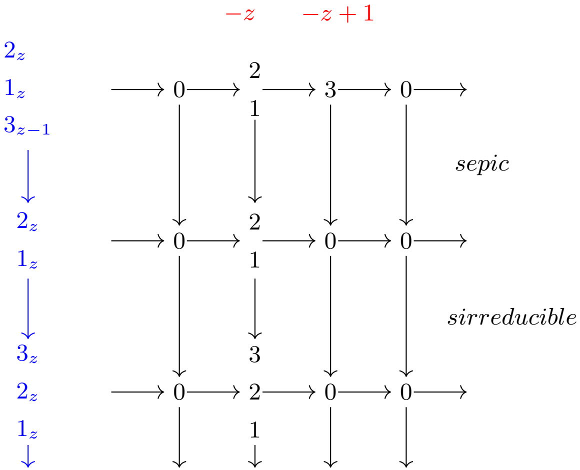
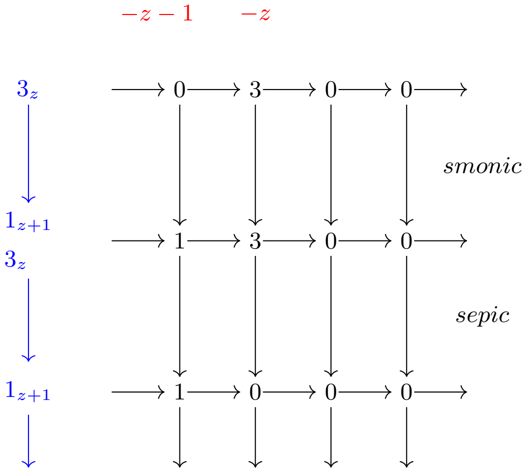
Auslander-Reiten triangle in $\widehat{A_1}$-mod



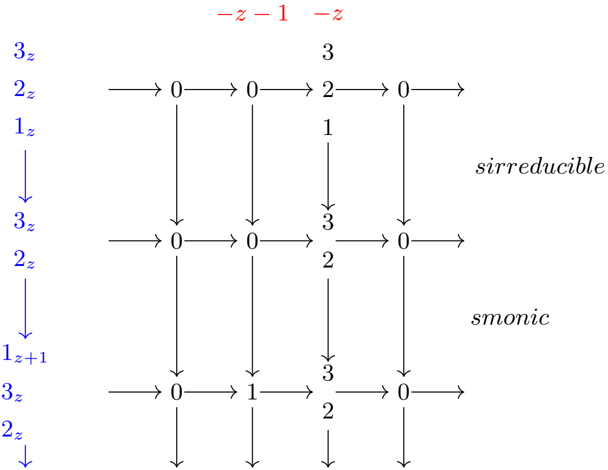
Auslander-Reiten triangle in $\widehat{A_1}$-mod



Auslander-Reiten triangle in $\widehat{A_1}$-mod

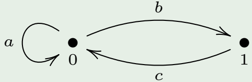
$$
\begin{array}{ccccccccccccc}
 & & & -z-1 & & -z & & & & & & & \\
1_z & & \longrightarrow & 0 & \longrightarrow & 1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & & \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & sirreducible \\
\begin{smallmatrix}2_z\\1_z\end{smallmatrix} & & \longrightarrow & 0 & \longrightarrow & \begin{smallmatrix}2\\1\end{smallmatrix} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & & \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & sirreducible \\
2_z & & \longrightarrow & 0 & \longrightarrow & 2 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & & \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & &
\end{array}
$$

Auslander-Reiten triangle in $\widehat{A_1}$-$\underline{\text{mod}}$

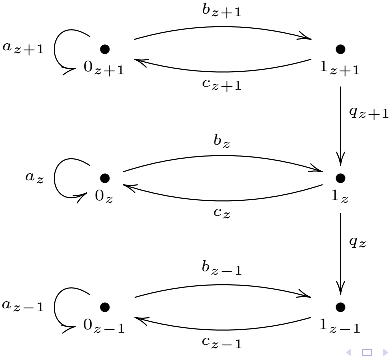


Example

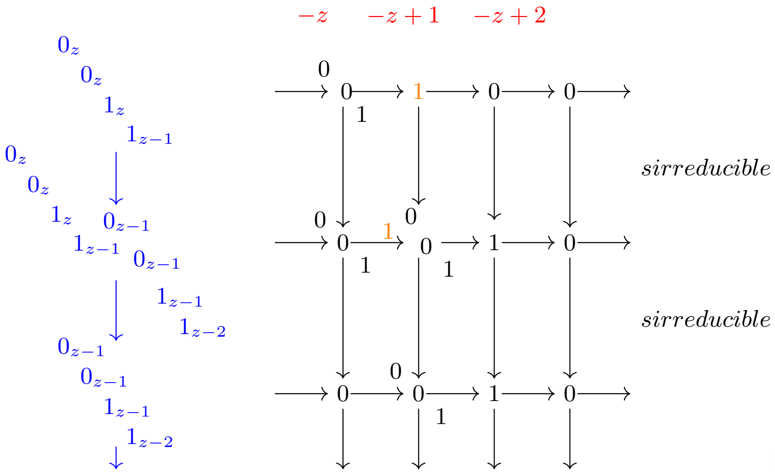
Let $A_2 = kQ/I$ be the finite dimensional algebra given by the quiver Q:=



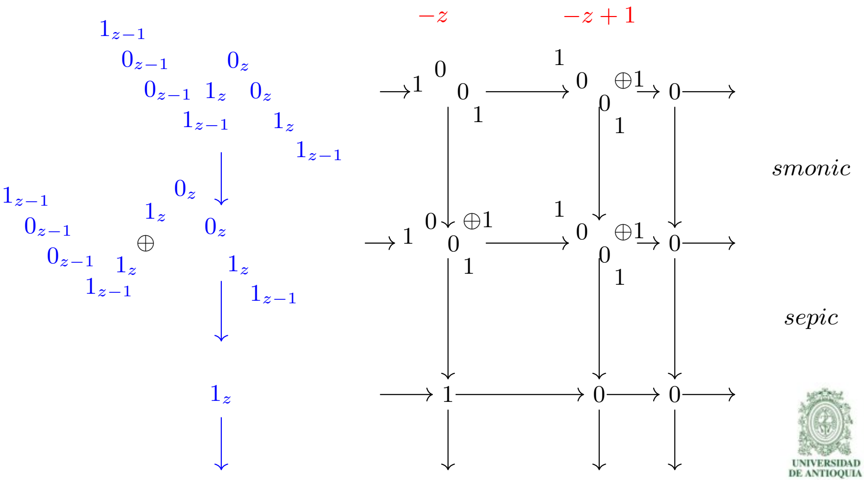
, where $I =< a^2, bc, cb >$. We have the quiver $\widehat{Q}$ is given by



Auslander-Reiten triangle in $\widehat{A_2}$-mod



Auslander-Reiten triangle in $\widehat{A_2}$-mod



Thanks

Section

1. Category of modules over repetitive algebras
2. Shapes of the irreducible morphisms
3. Shapes of Auslander-Reiten Triangles
4. Referencias

CALDERÓN-HENAO, Y., GIRALDO, H., AND VÉLEZ-MARULANDA, J. A.
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