Reverse plane partitions via representations of quivers

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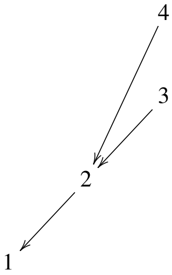
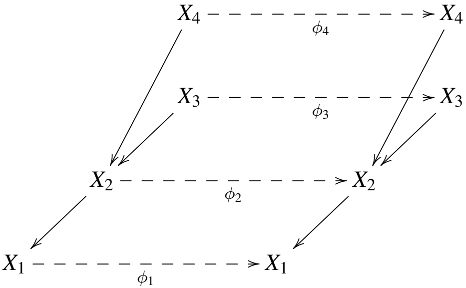
Outline

- nilpotent endomorphisms of quiver representations
- minuscule posets and Auslander–Reiten quivers
- reverse plane partitions on minuscule posets
- periodicity of promotion

- $\Lambda = \Bbbk Q/I$ - a finite dimensional algebra, $\overline{\Bbbk} = \Bbbk$
- $X = ((X_i)_i, (f_a)_a) \in \mathrm{rep}(Q, I) \simeq \mathrm{mod}\Lambda$
- $\phi = (\phi_i)_i$ a nilpotent endomorphism of X
- NEnd(X) - all nilpotent endomorphisms of X

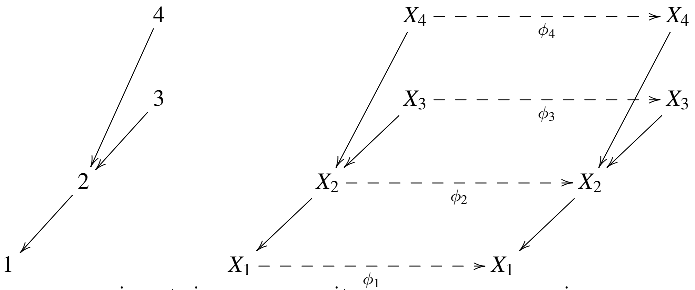


Q



Lemma

The space NEnd(X) is an irreducible algebraic variety.



For each i, $\phi_i \rightsquigarrow \lambda^i = (\lambda^i_1 \geqslant \cdots \geqslant \lambda^i_r)$ where partition λ^i records the sizes of the Jordan blocks of ϕ_i.

$$\mathrm{JF}(\phi) := (\lambda^1, \ldots, \lambda^n) \quad \text{the } \textbf{Jordan form data} \text{ of } \phi$$

For $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$ and $\lambda' = (\lambda'_1 \geqslant \cdots \geqslant \lambda'_{r'})$, one has $\lambda \leqslant \lambda'$ in **dominance order** if $\lambda_1 + \cdots + \lambda_\ell \leqslant \lambda'_1 + \cdots + \lambda'_\ell$ for each $\ell \geqslant 1$.

Theorem (G.–Patrias–Thomas, '18)

There is a unique maximum value of JF(·) on NEnd(X) with respect to componentwise dominance order, denoted by GenJF(X). It is attained on a dense open subset of NEnd(X).

Question

*For which subcategories $\mathcal{C}$ of rep(Q, I) is it the case that any object $X \in \mathcal{C}$ may be recovered from GenJF(X)? We say such a subcategory is **Jordan recoverable**.*

Example

Usually GenJF(X) is not enough information to recover X. Let $Q = 1 \leftarrow 2$.

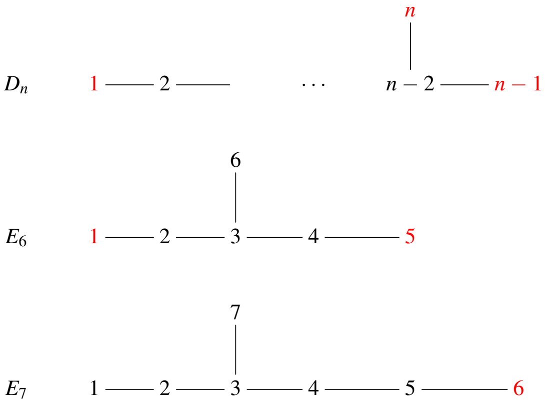
- $X = \Bbbk \xleftarrow{1} \Bbbk$ has $\mathrm{GenJF}(X) = ((1), (1))$
- $X' = \Bbbk \xleftarrow{0} \Bbbk$ has $\mathrm{GenJF}(X') = ((1), (1))$

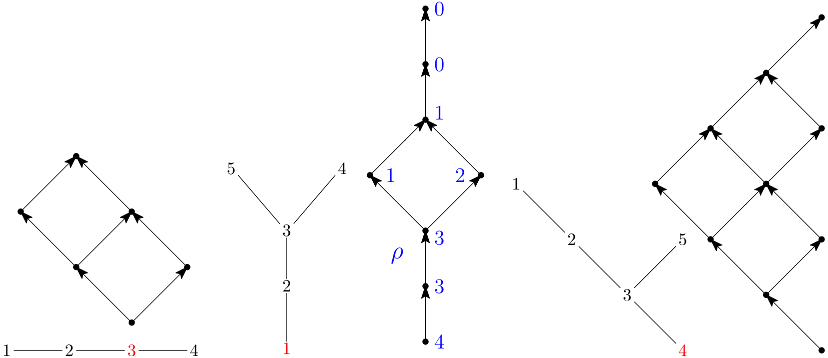
Theorem (G.–Patrias–Thomas ’18)

*Let Q be a Dynkin quiver and m a **minuscule vertex** of Q. The category $\mathcal{C}_{Q,m}$ of representations of Q all of whose indecomposable summands are supported at m is Jordan recoverable.*

Moreover, we classify the objects in $\mathcal{C}_{Q,m}$ in terms of the combinatorics of the **minuscule poset** associated with Q and m.

The minuscule posets are defined by choosing a simply-laced Dynkin diagram and a **minuscule vertex** m.



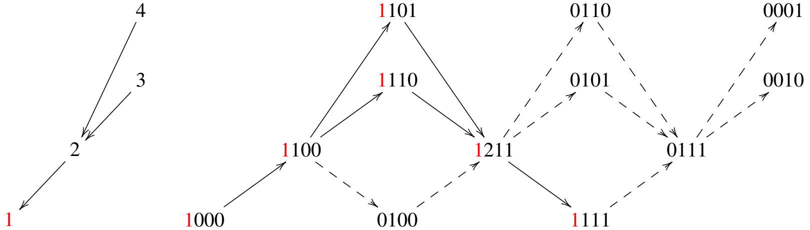


$\mathsf{P}_{A_4,3}$ $\mathsf{P}_{D_5,1}$ $\mathsf{P}_{D_5,4}$

A **reverse plane partition** is an order-reversing map $\rho : \mathsf{P} \to \mathbb{Z}_{\geqslant 0}$.
The objects of $\mathcal{C}_{Q,m}$ will be parameterized by **reverse plane partitions** defined on the minuscule poset associated with $\overline{Q}$ and m.

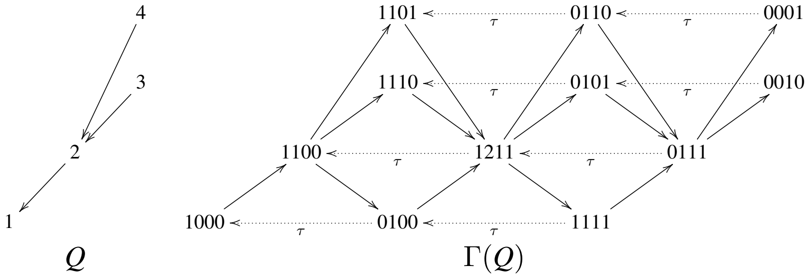
Lemma

Given a Dynkin quiver Q and a minuscule vertex m, the Hasse quiver of the minscule poset $\mathsf{P}_{\overline{Q},m}$ is isomorphic to the full subquiver of $\Gamma(Q)$ on the representations supported at m.



Q $\quad$ $\Gamma(Q)$ - the Auslander–Reiten quiver of Q

There is a map $\tau : \Gamma(Q)_0 \to \Gamma(Q)_0$ called the **Auslander–Reiten translation**.



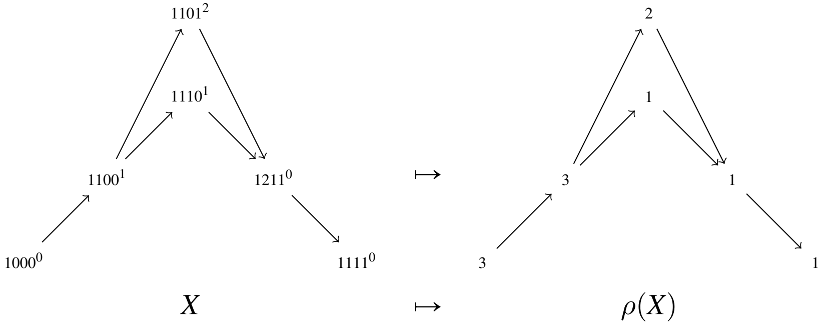
The Auslander–Reiten translation partitions the indecomposables into τ**-orbits**.

$$Q_0 \longleftrightarrow \{\tau\text{-orbits}\}$$

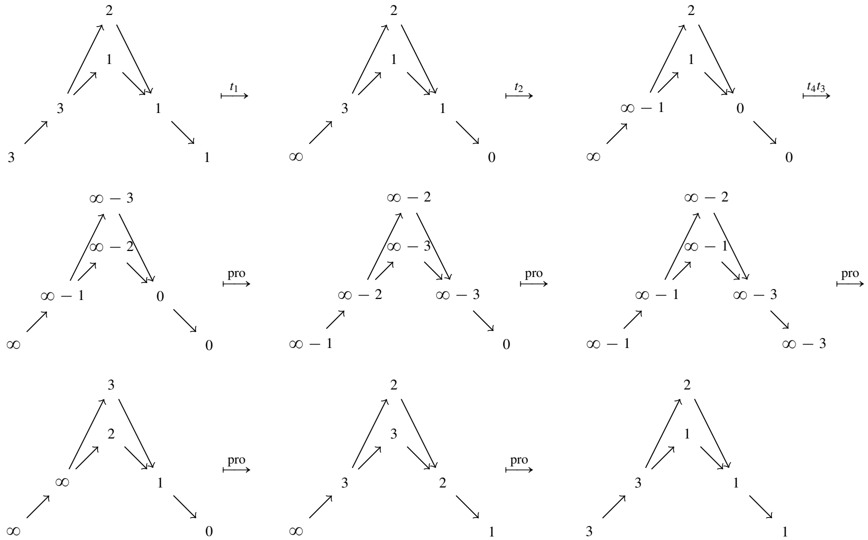
Theorem (G.–Patrias–Thomas '18)

The objects of $\mathcal{C}_{Q,m}$ are in bijection with RPP$(\mathsf{P}_{\overline{Q},m})$ via

$$X \quad \mapsto \quad \rho - \text{reverse plane partition from filling the } \tau\text{-orbits of } \mathsf{P}_{\overline{Q},m} \text{ with the Jordan block sizes in GenJF}(X)$$



Promotion (pro $= t_4 t_3 t_2 t_1$)



Theorem (G.–Patrias–Thomas '18)

We have $\mathrm{pro}^h = \mathrm{id}$ *where* h *is the Coxeter number of the root system.*

Let t_i be the operation of toggling every entry of $\rho \in RPP(\mathsf{P}_{\overline{Q},m})$ in τ-orbit i. Let pro $= t_n \cdots t_1$ where if $i, j \in Q_0$ and $i < j$, then there are no arrows $i \to j$.

Theorem (G.–Patrias–Thomas '18)

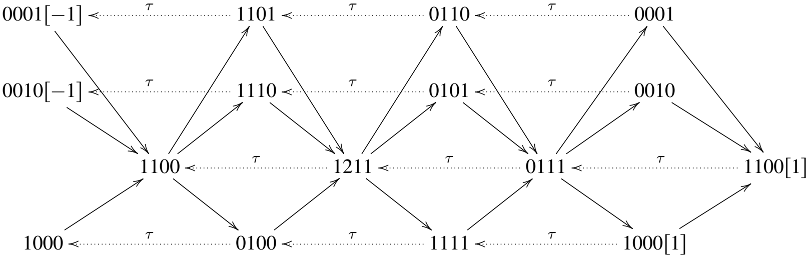
We have $pro^h = id$ *where* h *is the Coxeter number of the root system*

To prove this theorem, we interpret elements of $RPP(\mathsf{P}_{\overline{Q},m})$ with entries in

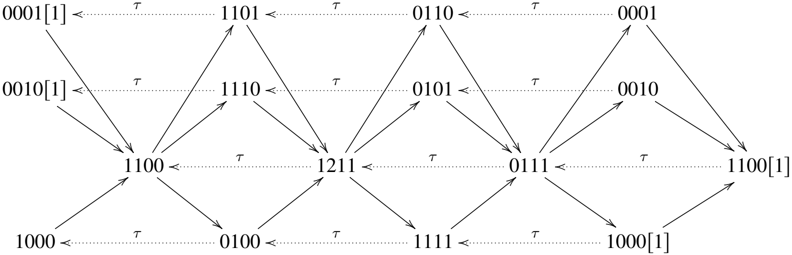
$$\{0, 1, \ldots, \infty - 1, \infty\}$$

as objects in the **root category** $\mathcal{R}_Q := D^b(Q)/[2]$.

- $D^b(Q)$ is the bounded derived category of Q
- its objects are cochain complexes of representations of Q up to quasi-isomorphisms
- the indecomposable objects of $\mathcal{R}_Q$ are indexed by roots in the associated root system



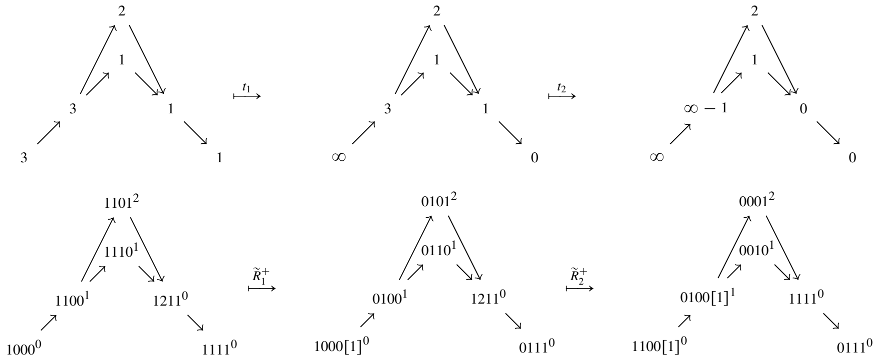
$\Gamma(D^b(Q))$ where $D^b(Q)$ is the bounded derived category of Q



$\Gamma(\mathcal{R}_Q)$ where $\mathcal{R}_Q := D^b(Q)/[2]$ is the root category

Theorem (G.–Patrias–Thomas ‘18)

For any $X \in \mathcal{C}_{Q,m} \subset \mathcal{R}_Q$, *one has* $t_i \rho(X) = \rho(\widetilde{R}_i^+(X))$.



The reflection functor $\widetilde{R}_i^+$ acts on dimension vectors as follows

$$\begin{array}{rcl} \widetilde{R}_i^+ : \mathbb{Z}^{|Q_0|} & \longrightarrow & \mathbb{Z}^{|Q_0|} \\ (x_1, \ldots, x_i, \ldots, x_n) & \longmapsto & (x_1, \ldots, -x_i + \sum_{k - i} x_k, \ldots, x_n) \\ \mathbf{dim}(X) & \longmapsto & \mathbf{dim}(\widetilde{R}_i^+(X)) \end{array}$$

if $X \in \text{rep}(Q)$ has no summands of $S(i)$.

Lemma (Gabriel '80)

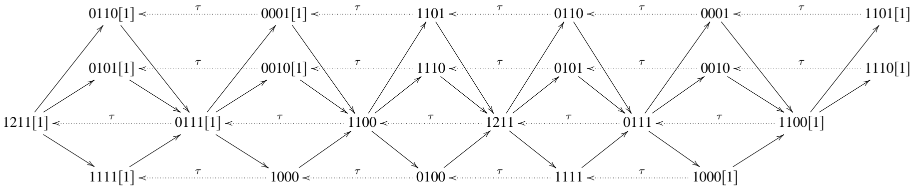
For any $X \in D^b(Q)$, $\tau(X) \simeq \widetilde{R}_n^+ \cdots \widetilde{R}_1^+(X)$.

Corollary

For any $X \in \mathcal{C}_{Q,m} \subset \mathcal{R}_Q$, *one has*

$$\rho(\tau(X)) = \rho(\widetilde{R}_n^+ \cdots \widetilde{R}_1^+(X)) = t_n \cdots t_1 \rho(X) = pro\,\rho(X).$$

In particular, $pro^h \rho(X) = \rho(\tau^h(X)) = \rho(X)$.



- Our periodicity statement in type A was previously established by Grinberg and Roby and from the tropical version of $A_{m-1} \times A_{n-m}$ Zamolochikov periodicity.

Thanks!