Dominant Dimension and Orders over Cohen-Macaulay Rings

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Last Year

Last Year

Recall from last year (!) that for a commutative Gorenstein ring, the cohomology annihilator ideal is

$$\mathrm{ca}(R) = \bigcap_{M \in \mathrm{MCM}(R)} \mathrm{ann}_R \underline{\mathrm{End}}_R(M)$$

- If R has finite global dimension, then $\mathrm{ca}(R) = R$.
- Under mild assumptions, $V(\mathrm{ca}(R)) = \mathrm{sing}(R)$.

Last Year

Theorem

If R is the complete local coordinate ring of a reduced curve singularity, then the cohomology annihilator ideal coincides with the conductor ideal.

- The conductor of R is the ideal $\{r \in R \ : \ r\bar{R} \subseteq R\}$ where $\bar{R}$ is the integral closure of R in its total quotient ring.
- The conductor is also equal to $\mathrm{ann}_R \underline{\mathrm{End}}_R(\bar{R})$.
- In our case, the normalization is a module finite R-algebra, it is maximal Cohen-Macaulay as an R-module, and it has finite global dimension.

Last Year

Let R be a Gorenstein ring, Λ be a noncommutative ring and $f : R \to \Lambda$ be a ring homomorphism.

- f is a split monomorphism,
- Λ is finitely generated as an R-module,
- Λ is maximal Cohen-Macaulay as an R-module,
- Λ has finite global dimension δ,

Theorem

With these assumptions, we have

$$[\mathrm{ann}_R \underline{\mathrm{End}}_R(\Lambda)]^{\delta+1} \subseteq \mathrm{ca}(R) \subseteq \mathrm{ann}_R \underline{\mathrm{End}}_R(\Lambda).$$

This Year

This year

Let R be a Gorenstein ring **of Krull dimension at most 2**, Λ be a noncommutative ring and $f : R \to \Lambda$ be a ring homomorphism.

- f is a split monomorphism,
- Λ is finitely generated as an R-module,
- Λ is maximal Cohen-Macaulay as an R-module,
- Λ has finite global dimension δ,
- $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ **has projective dimension n as a Λ-module**.

Theorem

With these assumptions, we have

$$[\mathrm{ann}_R \underline{\mathrm{End}}_R(\Lambda)]^{n+1} \subseteq \mathrm{ca}(R) \subseteq \mathrm{ann}_R \underline{\mathrm{End}}_R(\Lambda).$$

In particular, if Λ^* is projective, then

$$\mathrm{ca}(R) = \mathrm{ann}_R \underline{\mathrm{End}}_R(\Lambda).$$

Definitions and Notations

- R is a Cohen-Macaulay local ring with canonical module ω_R.
- Λ is an R-order. That is, it is a module-finite R-algebra which is maximal Cohen-Macaulay as an R-module.
- $\mathrm{MCM}(\Lambda) = \{X \in \Lambda\text{-mod} \ : \ X \in \mathrm{MCM}(R)\}$.
- $D = \mathrm{Hom}_R(-, \omega_R) : \mathrm{MCM}(\Lambda) \to \mathrm{MCM}(\Lambda^{\mathrm{op}})$ - it is an exact duality.
- $\omega_\Lambda = D\Lambda$ is the canonical module of Λ.

The following are equivalent [Iyama-Wemyss]:

1. ω_Λ is projective and Λ has finite global dimension,
2. Every maximal Cohen-Macaulay Λ-module is projective.
3. $\mathrm{gldim}\Lambda_p = \dim R_p$ for every prime ideal p of R.
4. $\mathrm{gldim}\Lambda_m = \dim R_m$ where m is the maximal ideal of R.

If Λ satisfies one of the above conditions, then it is called a *non-singular* order.

If ω_Λ is projective, then we have a version of Auslander-Buchsbaum formula:

$$\mathrm{pd}_\Lambda M + \mathrm{depth} M = \dim R$$

for any Λ-module M of finite projective dimension [Iyama-Reiten, Iyama-Wemyss]. [Josh Stangle] generalizes this in his PhD thesis as follows: If ω_Λ has projective dimension n, then

$$\dim R \leq \mathrm{pd}_\Lambda M + \mathrm{depth} M \leq n + \dim R$$

for every Λ-module M of finite projective dimension.

Question

- If Λ is non-singular, then every maximal Cohen-Macaulay module is projective.
- If Λ has finite global dimension with a canonical module ω_Λ of positive projective dimension, there are non-projective maximal Cohen-Macaulay modules.
- How do we understand the structure of the *stable* category of maximal Cohen-Macaulay modules?
- For instance, how many indecomposable non-projective maximal Cohen-Macaulay modules are there? (Auslander-Roggenkamp).

Injectives in $\mathrm{MCM}(R)$

- The canonical module ω_Λ is an injective object in $\mathrm{MCM}(\Lambda)$ and in fact any MCM-relatively injective Λ-module is isomorphic to a direct summand of finite direct sums of ω_Λ.
- The duality $D = \mathrm{Hom}_R(-, \omega_R)$ takes projectives to MCM-relatively injectives and vice versa.
- Dualizing a projective resolution of the maximal Cohen-Macaulay Λ^{op}-module DM gives a MCM-relatively injective coresolution of the maximal Cohen-Macaulay Λ-module M.
- The relative injective dimension of Λ is equal to the projective dimension of ω_Λ.

Dominant Dimension

Let $0 \to \Lambda \to I^0 \to I^1 \to \dots \to I^{k-1} \to I^k \to \dots$ be a minimal MCM-relatively injective coresolution of Λ. We say that Λ has MCM-relative dominant dimension at least k if $I^0, \dots, I^{k-1}$ are projective.

- If Λ is a non-singular order, then its MCM-relative dominant dimension is ∞.
- If Λ is an order of the form $\mathrm{End}_R(M)$ where $M \in \mathrm{MCM}(R)$, then the MCM-relative dominant dimension is at least $\max\{2, \dim R - 2\}$.
- If R is a regular local ring and Q is a linearly directed A_n quiver, then the path algebra RQ has MCM-relative dominant dimension 1.

Tilting

Let $0 \to \Lambda \to I^0 \to I^1 \to \ldots \to I^{k-1} \to I^k \to \ldots$ be a minimal MCM-relatively injective coresolution of Λ. Denote the image of $I^j \to I^{j+1}$ by K_{j+1}.

Lemma

Then, K_{j+1} is also a maximal Cohen-Macaulay module.

Theorem

If Λ has relative dominant dimension at least k and $j < k$, then the module

$$T_j = \bigoplus_{i=0}^{j} I^j \oplus K_{j+1}$$

is a k-tilting Λ-module.

Theorem

Let $\Gamma_j = \mathrm{End}_\Lambda(T_j)^{\mathrm{op}}$ where T_j is the tilting module defined above and suppose that Λ has finite global dimension. Then,

1. *Γ_j is also an R-order of finite global dimension.*
2. ****The projective dimension of ω_{Γ_j} is at most the projective dimension of ω_Λ.*

Note: See [Pressland, Sauter] and [Nguyen, Reiten, Todorov, Zhu] for the Artinian case.

THANK YOU!