Standard Conjecture D for Matrix Factorizations

Joint w/ Mark Walker

Goal of the project: settle a special case of Marcolli-Tabuada's noncomm. generalization of Grothendieck's Standard Conj. D.

Background: In the 60's, Grothendieck posed a family of conjectures concerning algebraic cycles called the Standard Conjectures. There are 5 of them: Conjectures A - D, and the Hodge Standard Conjecture (not to be confused w/ the Hodge Conjecture).

In this talk, I'll be interested in Conjecture D:

Conjecture D: Let X be a smooth projective complex variety. For vector bundles $\mathcal{F}, \mathcal{G}$ on X, define

$$\chi(\mathcal{F}, \mathcal{G}) = \sum_i (-1)^i \dim_{\mathbb{C}} \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}).$$

Then, $\chi(\mathcal{F}, -) = 0 \iff ch(\mathcal{F}) = 0$.

Note: the full conjecture is formulated for varieties over any base field; this is just the version over $\mathbb{C}$.

Conj. D is known when $\dim X \leq 4$ and when X is an abelian variety, but is wide open in general.

Marcolli-Tabuada posed an abstract, categorical generalization of Conj D.

I'll spend most of the talk establishing enough background to state MT's noncomm. Conj. D, and then I'll discuss some progress on the conjecture I've made w/ Mark Walker.

dg categories

A dg category $\mathcal{C}$ over $\mathbb{C}$ is:
- A set $\mathrm{Ob}(\mathcal{C})$ of objects,
- $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, a complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of $\mathbb{C}$-vect. spaces
- $\forall X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a composition morphism (of complexes)

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

Examples:

- dg cat w/ 1 object = dg a
- if R is a $\mathbb{C}$-alg (not necessarily comm.) $D^b(R)$, $\mathrm{Perf}(R)$ form dg cat's. Similarly for schemes.
- if R is a $\mathbb{C}$-alg, the singularity cat. of R forms a dg cat.

To formulate a dg categorical generalization of Conj. D, we need analogues of:
- singular cohomology
- Chern characters
- Euler pairings

Cohomology: we have a functor

$$HP_* : dg_{\mathbb{C}}\ cat \longrightarrow \text{graded } \mathbb{C}\text{-vect. spaces}$$

given by periodic cyclic homology.

Facts:

- $HP_i(\mathcal{C}) \cong HP_{i+2}(\mathcal{C}) \quad \forall i \in \mathbb{Z}$
- if X is smooth $/\mathbb{C}$, $HP_i(\mathrm{Perf}\,X) \cong \begin{cases} H^{even}(X;\mathbb{C}), & i \text{ even} \\ H^{odd}(X;\mathbb{C}), & i \text{ odd} \end{cases}$

Chern character: if $\mathcal{C}$ is a dg cat, and $X \in Ob(\mathcal{C})$, we have morphisms of dg cat's

$$\mathbb{C} \longrightarrow \mathrm{End}_{\mathcal{C}}(X) \longrightarrow \mathcal{C}$$

$$\begin{tikzcd} \rightsquigarrow & HP_0(\mathbb{C}) \arrow[r] & HP_0(\mathrm{End}_{\mathcal{C}}(X)) \arrow[r] & HP_0(\mathcal{C}) \\ & \mathbb{C} \arrow[u, "\cong"] \arrow[urr, dashed, "\Phi_X"'] & & \end{tikzcd}$$

Def'n: $ch(X) := \Phi_X(1)$.

Fact: this recovers the classical Chern character when $\mathcal{C} = \mathrm{Perf}(X)$, X a smooth proj. complex variety.

Euler Pairing: A dg cat $\mathcal{C}$ is proper if, $\forall X, Y \in \mathcal{C}$,

$$\sum_i \dim_{\mathbb{C}} H^i \mathrm{Hom}_{\mathcal{C}}(X,Y) < \infty.$$

$\mathrm{Perf}\,X$ is proper iff X is proper (if X is separated, Noetherian)

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If $\mathcal{C}$ is proper, define

$$\chi(X,Y) := \sum_i (-1)^i \dim_{\mathbb{C}} H^i \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

Also $\exists$ notion of <u>smoothness</u> for dg categories.

Perf X is smooth iff X is smooth$/_{\mathbb{C}}$ (X separated, finite type $/_{\mathbb{C}}$)

NC Standard Conj D (Marcolli-Tabuada, 2011) If $\mathcal{C}$ is a smooth and proper dg cat$/_{\mathbb{C}}$, and $X \in \mathcal{C}$, $\chi(X,-) = 0$ iff $ch(X) = 0$.

This recovers Grothendieck's Conj. D when $\mathcal{C} = \mathrm{Perf}\, X$, X smooth, projective$/_{\mathbb{C}}$.

R: complex hypersurface ring w/ isolated singularity.

$D_{sing}(R)$: singularity cat. of R.

Thm (B-Walker) NC Conj. D holds for $D_{sing}(R)$.

(this needs to be properly interpreted: $D_{sing}(R)$ is a differential $\mathbb{Z}/2$-graded category: the Hom complexes are 2-periodic. So it obviously isn't proper as a $\mathbb{Z}$-graded dg cat, but it <u>is</u> when considered as a $\mathbb{Z}/2$-graded cat. χ in this case is defined to be the mod 2 Euler characteristic.)

Question: if A is a finite-dim'l $\mathbb{C}$-alg of finite global dim, $D^b(A)$ is sm and proper. Does NC Conj. D hold for $D^b(A)$?