

CMness is determined by inertia groups

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Permutation invariants

$G \subset S_n$ acts on $\mathbb{Z}[x_1, \dots, x_n]$. Problem: describe invariant subring.

Theorem (Fundamental Theorem on Symmetric Polynomials)

If $G = S_n$, then

$$\mathbb{Z}[x_1, \dots, x_n]^G = \mathbb{Z}[\sigma_1, \dots, \sigma_n],$$

where $\sigma_1 = \sum_i x_i$, $\sigma_2 = \sum_{i < j} x_i x_j$, etc.

What if $G \subsetneq S_n$?

Permutation invariants

Over \mathbb{Q} :

Theorem (Kronecker 1881)

$\mathbb{Q}[x_1, \dots, x_n]^G$ is a free module over $\mathbb{Q}[\sigma_1, \dots, \sigma_n]$.

Kronecker's contribution is not well-known, but a modern invariant theorist would see this as an immediate consequence of the Hochster-Eagon theorem.

Example

$G = \langle (1234) \rangle \subset S_4$, acting on $\mathbb{Q}[x, y, z, w]$.

$$g_0 = 1$$

$$g_2 = xz + yw$$

$$g_3 = x^2y + y^2z + \dots$$

$$g_{4a} = x^2yz + y^2zw + \dots$$

$$g_{4b} = xy^2z + yz^2w + \dots$$

$$g_5 = x^2y^2z + y^2z^2w + \dots$$

is a basis over $\mathbb{Q}[\sigma_1, \dots, \sigma_4]$.

$$x^3y^2z + y^3z^2w + \dots = \frac{1}{2}\sigma_3g_3 - \frac{1}{2}\sigma_2g_{4b} + \frac{1}{2}\sigma_1g_5$$

Permutation invariants

Statement fails over \mathbb{Z} .

$$x^3y^2z + y^3z^2w + \cdots = \frac{1}{2}\sigma_3g_3 - \frac{1}{2}\sigma_2g_4b + \frac{1}{2}\sigma_1g_5$$

Problem

For which $G \subset S_n$ does the statement of Kronecker's theorem hold over \mathbb{Z} ?

Equivalent to:

Problem

For which $G \subset S_n$ is $k[x_1, \dots, x_n]^G$ a Cohen-Macaulay ring for any field k ?

Known as of 2016:

- If $k[x_1, \dots, x_n]^G$ is CM and $\text{char } k = p$, then G is generated by bireflections and p' -elements. (Gordeev & Kemper '03)
- If $G =$
 - S_n (classical)
 - A_n (classical)
 - $S_n \xrightarrow{\Delta} S_n \times S_n \subset S_{2n}$ (Reiner '90 / '03), or
 - $S_2 \wr S_n \subset S_{2n}$ (Hersh '03),

then $k[x_1, \dots, x_n]^G$ is CM regardless of k .

Permutation invariants

Let $k[\mathbf{x}] := k[x_1, \dots, x_n]$.

Theorem (BBS '17)

If $G \subset S_n$ is generated by transpositions, double transpositions, and 3-cycles, then

$$k[\mathbf{x}]^G$$

is Cohen-Macaulay regardless of k .

Theorem (BBS - Sophie Marques '18)

The converse is also true.

Compare: Chevalley-Shepard-Todd [cf. Ellen's talk]

Story of the “only-if” direction

The “if” direction used Stanley-Reisner theory to translate an orbifold result of Christian Lange (“topological Chevalley-Shephard-Todd”) into a CMness result for a certain Stanley-Reisner ring $k[\Delta/G]$, and then work of Garsia and Stanton '84 to translate this into the desired result for $k[\mathbf{x}]^G$.

By a topological argument, for the Stanley-Reisner ring $k[\Delta/G]$, the “only-if” held too. However, the arguments of Garsia-Stanton do not allow one to transfer this conclusion back to $k[\mathbf{x}]^G$.

After I defended, Sophie Marques proposed to transfer the *argument*, rather than the conclusion, from $k[\Delta]^G$ to $k[\mathbf{x}]^G$.

This necessitated a search for a commutative-algebraic fact to replace each topological fact we used.

Local structure in a quotient – slice theorem

Let X be a hausdorff topological space carrying an action by a finite group G . Let $x \in X$. Let G_x be the stabilizer of x for the action of G . Let X/G be the topological quotient, and let \bar{x} be the image of x in X/G .

Theorem (slice theorem for finite groups)

There is a neighborhood U of x , invariant under G_x , such that U/G_x is homeomorphic to a neighborhood of \bar{x} in X/G .

The name “slice theorem” comes from an analogous result for compact Lie groups.

Local structure in a quotient – slice theorem

What is the commutative-algebraic analogue?

There is an algebraic group analogue to the Lie group slice theorem called *Luna's étale slice theorem*, but for finite G , there is something much more general.

Let A be a ring with an action of a finite group G .

- $x \in X$ becomes $\mathfrak{P} \triangleleft A$.
- X/G becomes A^G .
- $\bar{x} \in X/G$ becomes $\mathfrak{p} = \mathfrak{P} \cap A^G$.
- $G_{\bar{x}}$ becomes $I_G(\mathfrak{P}) := \{g \in G : a - ga \in \mathfrak{P}, \forall a \in A\}$. (Not $D_G(\mathfrak{P})$!)
- The appropriate analogue for the sufficiently small neighborhood of \bar{x} in X/G is the *strict henselization* of A^G at \mathfrak{p} .

Local structure in a quotient – slice theorem

Let C be a (commutative, unital) ring. Let \mathfrak{p} be a prime ideal of C . The *strict henselization* of C at \mathfrak{p} is a local ring $C_{\mathfrak{p}}^{hs}$ together with a local map $C_{\mathfrak{p}} \rightarrow C_{\mathfrak{p}}^{hs}$ with the following properties:

- 1 $C_{\mathfrak{p}}^{hs}$ is a henselian ring.
- 2 $\kappa(C_{\mathfrak{p}}^{hs})$ is the separable closure of $\kappa(C_{\mathfrak{p}})$.
- 3 $C_{\mathfrak{p}}$ and $C_{\mathfrak{p}}^{hs}$ are simultaneously noetherian (resp. CM).
- 4 $C_{\mathfrak{p}} \rightarrow C_{\mathfrak{p}}^{hs}$ is faithfully flat of relative dimension zero.

$C_{\mathfrak{p}}^{hs}$ is universal with respect to 1 and 2. It should be viewed as a “very small neighborhood of \mathfrak{p} in C .”

Local structure in a quotient – slice theorem

Let A be a ring with an action by a finite group G . Let \mathfrak{p} be a prime of A^G . Let $C_{\mathfrak{p}}^{hs}$ be the strict henselization of A^G at \mathfrak{p} . Define

$$A_{\mathfrak{p}}^{hs} := A \otimes_{A^G} C_{\mathfrak{p}}^{hs}$$

Note G acts on $A_{\mathfrak{p}}^{hs}$ through its action on A .

Let \mathfrak{P} be a prime of A lying over \mathfrak{p} and let Ω be a prime of $A_{\mathfrak{p}}^{hs}$ pulling back to \mathfrak{P} . Recall $I_G(\mathfrak{P}) := \{g \in G : a - ga \in \mathfrak{P}, \forall a \in A\}$. (Fact: $I_G(\Omega) = I_G(\mathfrak{P})$.)

Theorem (Raynaud '70)

There is a ring isomorphism $(A_{\mathfrak{p}}^{hs})_{\Omega}^{I_G(\mathfrak{P})} \cong C_{\mathfrak{p}}^{hs}$.

This is the commutative-algebraic analogue!

Corollary (BBS - Marques '18)

Assume A^G is noetherian. Then TFAE:

- 1 A^G is CM.
- 2 For every prime \mathfrak{p} of A^G and every Ω of A_p^{hs} pulling back to a \mathfrak{P} of A lying over \mathfrak{p} ,

$$(A_p^{hs})_{\Omega}^{I_G(\mathfrak{P})}$$

is CM.

- 3 For every maximal \mathfrak{p} of A^G , there is some Ω of A_p^{hs} pulling back to a \mathfrak{P} of A lying over \mathfrak{p} , such that

$$(A_p^{hs})_{\Omega}^{I_G(\mathfrak{P})}$$

is CM.

Permutation invariants - “only-if” direction

Back to the permutation group context. Let $k[\mathbf{x}] = \mathbb{F}_p[x_1, \dots, x_n]$, for some prime p to be determined later.

Let N be the subgroup of G generated by transpositions, double transpositions, and 3-cycles.

Goal: prove that, if $N \subsetneq G$, for the right choice of p , $k[\mathbf{x}]^G$ is not CM.

By above, it suffices to find, when $N \subsetneq G$, a $p \triangleleft k[\mathbf{x}]^G$ such that the corresponding C_p^{hs} is not CM.

Permutation invariants - “only-if” direction

Note that G/N acts on $k[\mathbf{x}]^N$.

Theorem (BBS - Marques '18)

If there is a prime \mathfrak{P} of $k[\mathbf{x}]^N$ whose inertia group $I_{G/N}(\mathfrak{P})$ is a p -group, then $k[\mathbf{x}]^G$ is not CM.

(Recall $k = \mathbb{F}_p$.)

The main ingredients of the proof are:

- the above result which says that CMness at $\mathfrak{P} \cap k[\mathbf{x}]^G$ only depends on the action of $I_{G/N}(\mathfrak{P})$ on the appropriate strict henselization.
- a theorem of Lorenz and Pathak '01 which shows that such $I_{G/N}(\mathfrak{P})$ obstructs CMness.

It also uses the “if” direction to conclude that $k[\mathbf{x}]^N$ is CM. The invocation of Lorenz and Pathak needs this.

Permutation invariants - “only-if” direction

So the problem is reduced to finding a prime number p and a prime ideal \mathfrak{P} of $k[\mathbf{x}]^N$ such that $I_{G/N}(\mathfrak{P})$ is a p -group, when $N \subsetneq G$.

Let Π_n be the poset of partitions of $[n]$, ordered by refinement.

Each $\pi \in \Pi_n$ corresponds to the ideal \mathfrak{P}_π^* of $k[\mathbf{x}]$ generated by $x_i - x_j$ for each pair i, j in the same block of π . (Cf. the braid arrangement.)

Let G_π^B be the blockwise stabilizer of π in G , and let $G_\pi^B N/N$ be its image in G/N . If $\mathfrak{P}_\pi = \mathfrak{P}_\pi^* \cap k[\mathbf{x}]^N$, one can show that

$$I_{G/N}(\mathfrak{P}_\pi) = G_\pi^B N/N.$$

Permutation invariants - “only-if” direction

So we just need to find π such that $G_\pi^B N/N$ is a p -group.

Consider the map

$$\varphi : G \rightarrow \Pi_n$$

that sends a permutation g to the decomposition of $[n]$ into orbits of g .

Proposition (BBS '17)

If $g \in G \setminus N$ is such that $\pi = \varphi(g)$ is minimal in $\varphi(G \setminus N)$, then $G_\pi^B N/N$ has prime order (and is generated by the image of g).

If $N \subsetneq G$, $G \setminus N$ is nonempty, so such g exists, and fixing p as the order of $G_\pi^B N/N$, we find that $k[\mathbf{x}]^G$ is not CM. \square

Permutation invariants - further questions

- Is there a uniform proof of Lange's theorem?
- Is there a purely algebraic proof of the “if” direction?
- Given G not generated by transpositions, double transpositions, and three-cycles, find all the “bad” primes?

Thank you!