Tame quivers have finitely many m-maximal green sequences

Kiyoshi Igusa[1] Ying Zhou[2]

[1]Department of Mathematics
Brandeis University

[2]Department of Mathematics
Brandeis University

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Outline

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Tame Quivers
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Definition

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A *tame algebra* is a k-algebra such that for each dimension there are finitely many 1-parameter families that parametrize all but finitely many indecomposable modules of the algebra.

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A *tame algebra* is a k-algebra such that for each dimension there are finitely many 1-parameter families that parametrize all but finitely many indecomposable modules of the algebra.

Example

Here are all the (connected) tame quivers, $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6 . \tilde{E}_7, \tilde{E}_8$.

Tame Quivers

Auslander-Reiten Quivers

Theorem

The Auslander-Reiten quiver of a tame path algebra consists of three parts, the preprojectives, the preinjectives and the regulars.

Tame Quivers

Preprojective and preinjective components

Here are some basic properties of preprojective and preinjective components of AR quivers of basic tame hereditary algebras.

1. The AR quiver of kQ has one preprojective component which looks like $\mathbb{N}Q^{op}$

Tame Quivers
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2. The AR quiver of kQ has one preinjective component which looks like $-\mathbb{N}Q^{op}$.

Tame Quivers

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Here are some basic properties of preprojective and preinjective components of AR quivers of basic tame hereditary algebras.

1. The AR quiver of kQ has one preprojective component which looks like $\mathbb{N}Q^{op}$
2. The AR quiver of kQ has one preinjective component which looks like $-\mathbb{N}Q^{op}$.
3. All preprojective and preinjective modules in kQ are rigid.

Tame Quivers

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2. The AR quiver of kQ has one preinjective component which looks like $-\mathbb{N}Q^{op}$.
3. All preprojective and preinjective modules in kQ are rigid.
4. All but finitely many preprojectives and preinjectives are sincere.

Tame Quivers
Regular components

Here are some basic properties of regular components of AR quivers of basic tame hereditary algebras.

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Regular components

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1. There are infinitely many regular components, all of which are standard tubes $\mathbb{Z}A_\infty/(\tau^k)$.
2. All but at most three tubes have $k = 1$. In this case we consider the component homogeneous.
3. All elements in a homogeneous tube are non-rigid, hence they and their shifts can not be summands of any silting object.

Tame Quivers

Regular components

Here are some basic properties of regular components of AR quivers of basic tame hereditary algebras.

4. In a nonhomogeneous component $\mathbb{Z}A_\infty/(\tau^k)$ only indecomposables with quasi-length less than k are rigid. In other words there are only finitely many rigid indecomposables in any nonhomogeneous component.

Tame Quivers

Regular components

Here are some basic properties of regular components of AR quivers of basic tame hereditary algebras.

4. In a nonhomogeneous component $\mathbb{Z}A_\infty/(\tau^k)$ only indecomposables with quasi-length less than k are rigid. In other words there are only finitely many rigid indecomposables in any nonhomogeneous component.
5. Only finitely many regular indecomposable modules are rigid. Hence only finitely many regular indecomposables and their shifts can appear in an m-maximal green sequence.

Tame Quivers
Standard Stable Tubes

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \arrow[dr] & \\
M_{33} \arrow[ur] \arrow[dr] & & M_{13} \arrow[ur] \arrow[dr] & & M_{23} \arrow[ur] \arrow[dr] & & M_{33} \\
 & M_{12} \arrow[ur] \arrow[dr] & & M_{22} \arrow[ur] \arrow[dr] & & M_{32} \arrow[ur] \arrow[dr] & \\
M_1 \arrow[ur] & & M_2 \arrow[ur] & & M_3 \arrow[ur] & & M_1
\end{tikzcd}$$

This is a standard stable tube with rank 3. M_{ik} is rigid iff $k \leq 2$.

Tame Quivers
Standard Stable Tubes

$$
\begin{tikzcd}[column sep=small, row sep=small]
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 & M_{12} \arrow[ur] \arrow[dr] & & M_{22} \arrow[ur] \arrow[dr] & & M_{32} \arrow[ur] \arrow[dr] & \\
M_1 \arrow[ur] & & M_2 \arrow[ur] & & M_3 \arrow[ur] & & M_1
\end{tikzcd}
$$

This is a standard stable tube with rank 3. M_{ik} is rigid iff $k \leq 2$.

Here $M_{ik} = \begin{matrix} M_{i+k-1} \\ \cdots \\ M_{i+1} \\ M_i \end{matrix}$. We define the *quasi-length* of M_{ik} as k.

Tame Quivers

Standard Stable Tubes

Now let's see a homogeneous tube.

Tame Quivers
Standard Stable Tubes

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$$
\begin{tikzcd}
\cdots \arrow[d, bend left] \\
M_3 \arrow[u, bend left] \arrow[d, bend left] \\
M_2 \arrow[u, bend left] \arrow[d, bend left] \\
M \arrow[u, bend left]
\end{tikzcd}
$$

Note that nothing in this tube is rigid.

Tame Quivers
Standard Stable Tubes

Now let's see a homogeneous tube.

$$\begin{tikzcd} \cdots \arrow[d, bend left] \\ M_3 \arrow[u, bend left] \arrow[d, bend left] \\ M_2 \arrow[u, bend left] \arrow[d, bend left] \\ M \arrow[u, bend left] \end{tikzcd}$$

Note that nothing in this tube is rigid.

M

Here $M_k = \ldots$

M

Tame Quivers

Example: $\tilde{D}_4$

So here is what an AR quiver of a tame path algebra looks like. In this example the quiver is

$$\begin{array}{ccccc} & & 2 & & \\ & & \downarrow & & \\ 1 & \longrightarrow & 5 & \longleftarrow & 3. \\ & & \uparrow & & \\ & & 4 & & \end{array}$$

Tame Quivers

Example: $\tilde{D}_4$-preprojectives

Here is the preprojective component, $\mathcal{P}$.

Tame Quivers

Example: $\tilde{D}_4$-preprojectives

Here is the preprojective component, $\mathcal{P}$.

$$\begin{tikzcd}
 & P_1 \arrow[ddr] & & \tau^{-1}P_1 \arrow[ddr] & & \cdots \\
 & P_2 \arrow[dr] & & \tau^{-1}P_2 \arrow[dr] & & \cdots \\
P_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau^{-1}P_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & \tau^{-2}P_5 \cdots & \\
 & P_3 \arrow[ur] & & \tau^{-1}P_3 \arrow[ur] & & \cdots \\
 & P_4 \arrow[uur] & & \tau^{-1}P_4 \arrow[uur] & & \cdots
\end{tikzcd}$$

Tame Quivers

Example: $\tilde{D}_4$-preinjectives

Here is the preinjective component, $\mathcal{Q}$.

Tame Quivers
Example: $\tilde{D}_4$-preinjectives

Here is the preinjective component, $\mathcal{Q}$.

$$
\begin{tikzcd}
\cdots & \tau I_1 \arrow[ddr] & & I_1 \\
\cdots & \tau I_2 \arrow[dr] & & I_2 \\
\cdots \tau I_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & & I_5 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \\
\cdots & \tau I_3 \arrow[ur] & & I_3 \\
\cdots & \tau I_4 \arrow[uur] & & I_4
\end{tikzcd}
$$

Tame Quivers

Example: $\tilde{D}_4$-regulars

Here are the regular components. There are infinitely many homogeneous tubes and 3 nonhomogeneous ones. All objects in the homogeneous ones are non-rigid. The quasi-simple in the homogeneous tubes has dimension vector is (1,1,1,1,2). The quasi-simples in the three nonhomogeneous tubes have dimension vectors (1,1,0,0,1) and (0,0,1,1,1), (1,0,1,0,1) and (0,1,0,1,1), (1,0,0,1,1) and (0,1,1,0,1) respectively.

Tame Quivers

AR quivers of Bounded Derived Categories

For a tame quiver Q there are infinitely many components of $D^b(kQ)$ consisting of shifts of preprojectives and preinjectives that are in the form $\mathbb{Z}Q^{op}$. Let's label these components *transjective*. The transjective component containing $\Lambda[m]$ is labelled $\mathcal{P}_m$.

Tame Quivers

AR quivers of Bounded Derived Categories

For a tame quiver Q there are infinitely many components of $D^b(kQ)$ consisting of shifts of preprojectives and preinjectives that are in the form $\mathbb{Z}Q^{op}$. Let's label these components *transjective*. The transjective component containing $\Lambda[m]$ is labelled $\mathcal{P}_m$. There are also infinitely many regular components. There are at most 3 nonhomogeneous tubes in $modkQ[m]$ for any m. There are also infinitely many homogeneous tubes in $modkQ[m]$ for any m. However since nothing in a homogeneous tube is rigid they don't affect our problem.

Outline

1. Background
   - Tame Quivers
   - Silting Objects
   - m-maximal green sequences
2. Our Result
   - Theorem
   - The proof

Silting Objects
Silting Objects

Definition

Let Λ be an algebra with n primitive idempotents. A *silting object* T of $D^b(\Lambda)$ is an object such that T has n direct summands and $(T, T[m]) = 0$ for all $m > 0$. A *pre-silting object* is an object that only has to satisfy the second condition.

Silting Objects

Ex: $D^b(A_3)$

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

Silting Objects
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P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

$\Lambda[i]$ is a silting object for any i. $T_1 = P_3[1] \oplus P_1[1] \oplus I_1[1]$ is also a silting object.

Silting Objects
Approximations

Definition

Let $\mathcal{C}$ be a category and $\mathcal{X}$ be one of its subcategories. If $M \in Ob\mathcal{C}, N \in Ob\mathcal{X}$, a morphism $f \in Hom_{\mathcal{C}}(M, N)$ is a *minimal left-$\mathcal{X}$ approximation* if for any $g \in End_{\mathcal{C}}N$ such that $g \circ f = f$ g is an isomorphism and for any $N' \in Ob\mathcal{X}$ for any $q \in Hom_{\mathcal{C}}(M, N')$ we have q factors through f.

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$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "q"'] & N \arrow[d, dashed, "l"] \\ & N' \end{tikzcd}$$

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$$\begin{tikzcd} M \arrow[r, "f"] & N \\ M' \arrow[u, dashed, "l"] \arrow[ur, "q"] & \end{tikzcd}$$

Silting Objects

Mutations

Definition

A *forward mutation* on the direct summand T_i of the silting object T is $T_i' \oplus (T/T_i)$ where T_i' is the cone/homotopy cokernel of the minimal left-$add(T/T_i)$ approximation of T_i.

Silting Objects
Mutations

Definition

A *forward mutation* on the direct summand T_i of the silting object T is $T_i' \oplus (T/T_i)$ where T_i' is the cone/homotopy cokernel of the minimal left-$add(T/T_i)$ approximation of T_i.
A *backward mutation* on the direct summand T_i of the silting object T is $T_i' \oplus (T/T_i)$ where T_i' is homotopy kernel/ [-1] of the cone/ of the minimal right-$add(T/T_i)$ approximation of T_i.

Silting Objects
Ex: $D^b(A_3)$

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

Silting Objects

Ex: $D^b(A_3)$

$$
\begin{tikzcd}
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& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}
$$

Λ is a silting object. When we do a forward mutation at P_3 we get $T' = S_2 \oplus P_2 \oplus P_1$. When we do a forward mutation at P_1 now we get $T'' = S_2 \oplus P_2 \oplus P_1[1]$. When we do another forward mutation at P_2 we get $T''' = S_2 \oplus P_3[1] \oplus P_1[1]$.

Outline

m-maximal green sequences

Definition

Definition

An *m-maximal green sequence* is a finite sequence of forward mutations from Λ to $\Lambda[m]$.

m-maximal green sequences

Example

$$\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}$$

m-maximal green sequences
Example

$$
\begin{tikzcd}
I_1[-1] \arrow[dr] & & P_1 \arrow[dr] & & P_3[1] \arrow[dr] & & S_2[1] \arrow[dr] & & I_1[1] \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ur] \arrow[dr] & & P_2[1] \arrow[ur] \arrow[dr] & & I_2[1] \arrow[ur] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & P_3[2]
\end{tikzcd}
$$

So $(P_1, P_2, P_3, P_1[1], P_2[1], P_3[1])$ is a 2-maximal green sequence.

Outline

Theorem
Theorem

Theorem

Tame quivers accept finitely many m-maximal green sequences.

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2. Our Result
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The proof

Basic ideas-Reducing the problem to indecomposable summands

The basic idea here is that if only finitely many indecomposable summands can appear in any m-maximal green sequence then only finitely many silting objects can appear in any m-maximal green sequence.

The proof

Basic ideas-Reducing the problem to indecomposable summands

The basic idea here is that if only finitely many indecomposable summands can appear in any m-maximal green sequence then only finitely many silting objects can appear in any m-maximal green sequence.When that happens only finitely many m-maximal green sequences can exist because a green sequence can not repeat silting objects due to Theorem 2.11 in [1].

The proof

Basic ideas-Transjectives and Regulars

Hence the problem has been reduced to the problem of whether there are only finitely many indecomposable summands of silting objects in any m-maximal green sequence.

The proof

Basic ideas-Transjectives and Regulars

Hence the problem has been reduced to the problem of whether there are only finitely many indecomposable summands of silting objects in any m-maximal green sequence.
There are only two kinds of indecomposable summands, namely transjectives of the form $\tau^i P_j[k]$ and regulars (i.e. shifts of regular modules).

The proof

Basic ideas-Regulars

However there are only finitely many nonhomogeneous regular components between Λ and $\Lambda[m]$ and each of them only have finitely many rigid indecomposables. Furthermore homogeneous regular components do not have any rigid objects. Hence only finitely many rigid regular indecomposable summands can appear in any m-maximal green sequence.

The proof
Basic ideas-Lemma 1

Hence the problem is really only about transjectives.

The proof

Basic ideas-Lemma 1

Hence the problem is really only about transjectives.We will follow the approach of Brustle-Dupont-Perotin here. We first prove that for a tame quiver with n vertices there are at most $n-2$ regulars in a silting object.

The proof

Basic ideas-Lemma 1

Hence the problem is really only about transjectives.We will follow the approach of Brustle-Dupont-Perotin here. We first prove that for a tame quiver with n vertices there are at most $n-2$ regulars in a silting object.

So here is our first lemma.

Lemma

1. *If Q is a tame quiver with n vertices there are at most $n-2$ regular indecomposable summands in a silting object of $D^b(kQ)$.*

The proof
Basic ideas-Lemma 2

Using properties of tame quivers it is easy to see using a type by type argument that Lemma 1 can be reduced to Lemma 2.

The proof

Basic ideas-Lemma 2

Using properties of tame quivers it is easy to see using a type by type argument that Lemma 1 can be reduced to Lemma 2.

Lemma

2. *If Q is a tame quiver with n vertices and $\mathcal{R}$ is one of the nonhomogeneous regular components of modkQ with k quasi-simples. Then at most $k-1$ summands in $\cup\mathcal{R}[m]$ of may appear in any silting object in $D^b(kQ)$.*

The proof

Silting compatibility

If $\{M_i\}_{i\in I}$ are a family of indecomposable modules of kQ and $\Pi_{i\in I} M_i[n_i]$ is not pre-silting for any $\{n_i\}_{i\in I}$ we say that $\{M_i\}_{i\in I}$ is *silting-incompatible*. Otherwise we say that it is *silting-compatible*.

The proof

Basic ideas-Lemma 2

Now we need two more short lemmas, Lemma 3 and Lemma 4, to prove Lemma 2.

Lemma

3. *If M and N are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of kQ. If $Hom(M,N) \neq 0$ and $Ext^1(N,M) \neq 0$, then M and N are silting-incompatible.*

The proof

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Now we need two more short lemmas, Lemma 3 and Lemma 4, to prove Lemma 2.

Lemma

3. *If M and N are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of kQ. If $Hom(M, N) \neq 0$ and $Ext^1(N, M) \neq 0$, then M and N are silting-incompatible.*
4. *If $M_1, \cdots, M_k$ are regular modules in a nonhomogeneous tube in the Auslander-Reiten quiver of kQ. If $Ext^1(M_i, M_{i+1}) \neq 0$ for any $1 \leq i < k$ and $Ext^1(M_k, M_1) \neq 0$, then $\{M_i\}$ is silting-incompatible.*

The proof

Basic ideas-Lemma 2

Proof.

For Lemma 3 since $Hom(M, N) \neq 0$, $Ext^{i-j}(M[i], N[j]) \neq 0$ if $i > j$. Since $Ext^1(N, M) \neq 0$ $Ext^{j-i+1}(N[j], M[i]) \neq 0$ if $i \leq j$. Hence $M[i] \oplus N[j]$ is not pre-silting for any arbitrary i and j. For Lemma 4 for arbitrary $n_1, \cdots n_k$ use the argument above it is easy to see that if $\oplus_{i=1}^{k} M_i[n_i]$ is pre-silting, then $n_2 > n_1$, $n_3 > n_2$, $\cdots, n_1 > n_k$ which is impossible. Hence $\{M_i\}$ is silting-incompatible. □

The proof
Basic ideas-Lemma 2

Now let's prove Lemma 2.

Proof.

If we assume that the conclusion in Lemma 2 is wrong we will reach a silting-incompatible scenario in either Lemma 3 or Lemma 4. Hence Lemma 2 is proven. □

The proof

Basic ideas-The degree argument

To make arguments easer we define the *transjective degree* of the transjective object $\tau^i P_j[k]$ to be i.
After proving Lemma 2 we can prove that any mutation that changes components can only change the transjective degree of a transjective object by some bounded amount. There are only finitely many possible green mutations that change components in any m-maximal green sequence. We can prove that any transjective indecomposable summand allowed in an m-maximal green sequence has bounded transjective degree.

The proof

Basic ideas-The degree argument

Lemma

([2], Lemma 10.1) Let H be a representation-infinite connected hereditary algebra. Then there exists $N \geq 0$ such that for any $k \geq N$, for any projective H-module P, the H-modules $\tau^{-k}P$ and $\tau^{k}P[1]$ are sincere.

The proof

Basic ideas-The degree argument

Lemma

([2], Lemma 10.1) Let H be a representation-infinite connected hereditary algebra. Then there exists $N \geq 0$ such that for any $k \geq N$, for any projective H-module P, the H-modules $\tau^{-k}P$ and $\tau^k P[1]$ are sincere.

Lemma

([2]) Let Q be a tame quiver and M_1, M_2 two transjective modules of kQ. If $\{M_1, M_2\}$ is silting-compatible, then $|deg(M_1) - deg(M_2)| \leq N$

The proof

Basic ideas-The degree argument

Proof.

If $k-l>N$ we need to prove that $\tau^k P_a$ and $\tau^l P_b$ are silting-incompatible. If $i \leq j$ $Ext^{j-i+1}(\tau^l P_b[j], \tau^k P_a[i]) = Ext^1(\tau^l P_b, \tau^k P_a) = Hom(\tau^{k-1} P_a, \tau^l P_b) = Hom(P_a, \tau^{l-k+1} P_b) \neq 0$ since $\tau^{l-k+1} P_b$ is a sincere preprojective module. If $i>j$ $Ext^{i-j}(\tau^k P_a[i], \tau^l P_b[j]) = Ext^1(\tau^k P_a[1], \tau^l P_b) = Hom(\tau^{l-1} P_a, \tau^k P_b[1]) = Hom(P_a, \tau^{k-l+1} P_b[1]) \neq 0$ since $\tau^{k-l+1} P_b[1]$ is a sincere preinjective module. Hence $\tau^k P_a$ and $\tau^l P_b$ are silting-incompatible. Exchange the objects if $k-l<-N$. Hence the lemma has been proven. □

The proof

Basic ideas-The degree argument

Now we only need to prove that the minimal transjective degree L of silting objects that can appear in m-maximal green sequences has a lower bound.

The proof

Basic ideas-The degree argument

Now we only need to prove that the minimal transjective degree L of silting objects that can appear in m-maximal green sequences has a lower bound.
Note that due to Lemma 1 there are at least 2 transjective components in any silting object in $D^b(kQ)$. No green mutation within a component or green mutation from a regular component to another one can increase L. All other green mutations may increase L by at most N.

The proof

Basic ideas-The degree argument

Now we only need to prove that the minimal transjective degree L of silting objects that can appear in m-maximal green sequences has a lower bound.
Note that due to Lemma 1 there are at least 2 transjective components in any silting object in $D^b(kQ)$. No green mutation within a component or green mutation from a regular component to another one can increase L. All other green mutations may increase L by at most N.
However there are only n summands of a silting object, $m+1$ transjective components and m regular components so the amount of mutations that can increase L is at most $2mn$. To reach $\Lambda[m]$ which is of degree 0 L has to always be at least $-2mnN$.

The proof

Basic ideas-The degree argument

Similarly we have an upper bound of maximal transjectve degree H of silting objects that can appear in m-maximal green sequences, namely $2mnN$. Hence any indecomposable transjective summand that can appear in m-maximal green sequences has transjective degree between $-2mnN$ and $2mnN$.

The proof

Basic ideas-The degree argument

Similarly we have an upper bound of maximal transjectve degree H of silting objects that can appear in m-maximal green sequences, namely $2mnN$. Hence any indecomposable transjective summand that can appear in m-maximal green sequences has transjective degree between $-2mnN$ and $2mnN$.
There are finitely many indecomposable transjective summands that can appear in m-maximal green sequences. Hence there are finitely many indecomposable summands that can appear in m-maximal green sequences which implies that tame quivers admit finitely many m-maximal green sequences.

Summary

- We proved that tame quivers have finitely many m-maximal green sequences.

For Further Reading I

- Takuma Aihara and Osamu Iyama, *Silting mutation in triangulated categories*, J London Math Soc (2012) 85 (3): 633-668.
- Thomas Brüstle, Grégoire Dupont and Matthieu Pérotin, On Maximal Green Sequences, arXiv:1205.2050 [math.RT], 2013.