

# DERIVED TAME NAKAYAMA ALGEBRAS

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JOINT-WORK WITH

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- Unless explicitly stated, all modules are finitely generated and from the left side.



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- For all projective  $\Lambda$ -modules  $P$ , let  $\mathbf{r}(P) = (p_1, p_2, \dots, p_n)$ , where  $p_1, p_2, \dots, p_n$  are non-negative integers such that

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- For all complexes of projective  $\Lambda$ -modules  $(P^\bullet, \delta^\bullet)$ , the **vector rank**  $\mathbf{r}^\bullet(P^\bullet)$  is defined as

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- A **rational family** of bounded minimal complexes over  $\Lambda$  is a bounded complex  $(P^\bullet, \delta^\bullet)$  of projective  $\Lambda$ - $R$ -bimodules, where  $R = \mathbb{k}[t, f(t)^{-1}]$  with  $f$  a non-zero polynomial, and  $\text{Im } \delta^n \subseteq \text{rad } P^{n+1}$ .



- For a rational family  $(P^\bullet, \delta^\bullet)$ , we define the complex  $P^\bullet(m, \lambda) = (P^\bullet \otimes_{\mathbb{k}} R/(t - \lambda)^m, \delta^\bullet \otimes 1)$  of projective  $\Lambda$ -modules, where  $m \in \mathbb{N}$ ,  $\lambda \in \mathbb{k}$ ,  $f(\lambda) \neq 0$ .

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**Definition 2** ((V. BEKKERT, YU. DROZD, ARXIV:MATH/0310352)). We say that  $\Lambda$  is **derived tame** if there exists a set  $\mathfrak{P}$  of rational families of bounded complexes over  $\Lambda$  such that:

- (i) For each bounded vector  $\mathbf{v}^\bullet = (\mathbf{v}_i)_{i \in \mathbb{Z}}$  of non-negative integers, the set  $\mathfrak{P}(\mathbf{v}^\bullet) = \{P^\bullet \in \mathfrak{P} \mid \mathbf{r}^\bullet(P^\bullet) = \mathbf{v}^\bullet\}$  is finite.

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**Theorem 4** ((CH. GEISS, H. KRAUSE, 2002)). *Derived tameness is preserved by derived equivalence.*

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**Definition 5.** Assume that  $\Lambda$  has finite global dimension. The **Euler form**  $\chi_\Lambda$  of  $\Lambda$  is defined on the Grothendieck group of  $\Lambda$  by

$$\chi_\Lambda(\mathbf{dim} M) = \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{k}} \operatorname{Ext}_{\Lambda}^i(M, M)$$

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- If  $\Lambda$  is either gentle or skewed-gentle, then  $\Lambda$  is derived tame (V. BEKKERT, H. MERKLEN, 2003) AND (V. BEKKERT, E. N. MARCOS, H. MERKLEN, 2003).



# DERIVED WILD ALGEBRAS AND THE DERIVED TAME-WILD DICHOTOMY THEOREM

**Definition 6** ((V. BEKKERT, YU. DROZD, ARXIV:MATH/0310352)). Let  $\Sigma = \mathbb{k}\langle x, y \rangle$ . We say that  $\Lambda$  is *derived wild* if there exists a bounded complex  $(P^\bullet, \delta^\bullet)$  of projective  $\Lambda$ - $\Sigma$ -bimodules such that  $\text{Im } \delta^n \subseteq \text{rad } P_{n+1}$ , and for all  $\Sigma$ -modules  $L$  and  $L'$  with finite dimension over  $\mathbb{k}$ , we have:

- (i)  $P^\bullet \otimes_\Sigma L \cong P^\bullet \otimes_\Sigma L'$  if and only if  $L \cong L'$ ;
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**Question:** Which Nakayama algebras  $\Lambda$  are derived tame?



Recall that  $\Lambda$  is said to be a **Nakayama algebra** if every left or right indecomposable projective  $\Lambda$ -module has a unique composition series.



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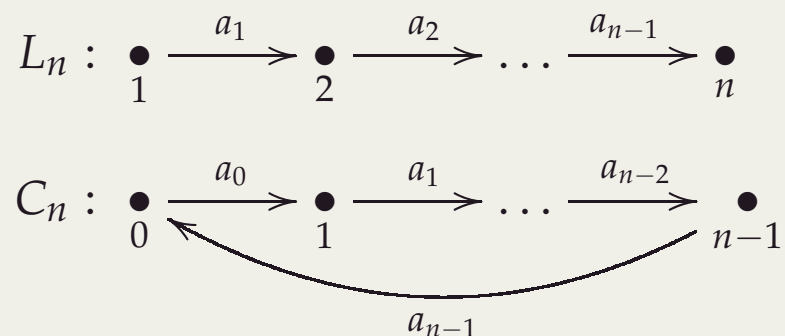
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**Theorem 8.**  $\Lambda$  is a Nakayama algebra if and only if  $\Lambda = \mathbb{k}Q/I$ , where  $Q$  is one of the following quivers:



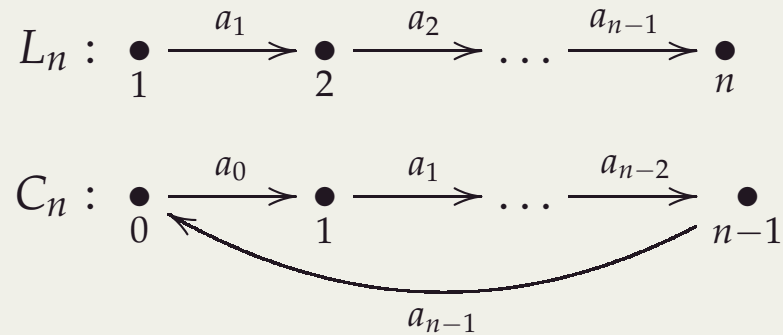
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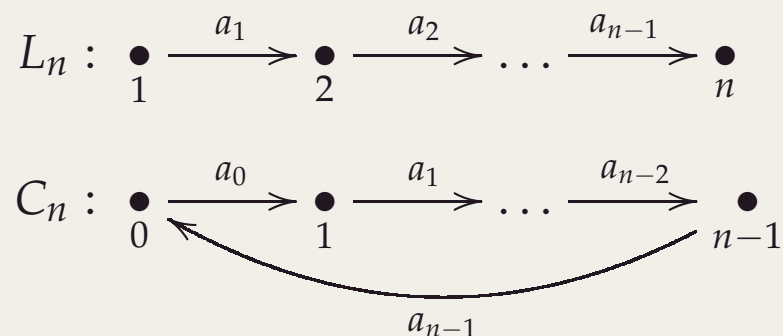
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- If  $Q = L_n$ , then we say that  $\Lambda$  is a **line algebra**.
- If  $Q = C_n$ , then we say that  $\Lambda$  is a **cycle algebra**.

# DERIVED TAME NAKAYAMA ALGEBRAS AND GORENSTEIN-PROJECTIVE MODULES

**Theorem 9.** (V. BEKKERT, H. GIRALDO, V-M, IN PROGRESS) *Assume that  $\Lambda$  is a Nakayama algebra. Then  $\Lambda$  is derived tame if and only if one of the following conditions holds:*

- (i)  *$\Lambda$  is a line algebra whose Euler form is non-negative.*
- (ii)  *$\Lambda$  is either gentle or derived equivalent to some skewed-gentle algebra.*

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**Definition 10.** (E. ENOCHS, O. JENDA, 1995) A  $\Lambda$ -module  $V$  is said to be **Gorenstein-projective** if there exists an acyclic complex of projective  $\Lambda$ -modules

$$P^\bullet : \dots \rightarrow P^{-2} \xrightarrow{\delta^{-2}} P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\delta^0} P^1 \xrightarrow{\delta^1} P^2 \rightarrow \dots$$

such that  $\text{Hom}_\Lambda(P^\bullet, \Lambda)$  is also acyclic and  $V = \text{coker } \delta^0$ . We denote by  $\Lambda\text{-Gproj}$  the category of Gorenstein-projective  $\Lambda$ -modules that are finitely generated, and by  $\underline{\Lambda\text{-Gproj}}$  its stable category.

**Definition 11.** The **singularity category** of  $\Lambda$  is defined to be the Verdier quotient

$$\mathcal{D}_{\text{sg}}(\Lambda\text{-mod}) = \mathcal{D}^b(\Lambda\text{-mod}) / \mathcal{K}^b(\Lambda\text{-proj}).$$



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- If  $\Lambda$  is Gorenstein, then  $\mathcal{D}_{\text{sg}}(\Lambda\text{-mod}) = \Lambda\text{-}\underline{\text{Gproj}}$  ((R.O. BUCHWEITZ, PREPRINT) and (D. HAPPEL, 1992)).

**Definition 11.** The **singularity category** of  $\Lambda$  is defined to be the Verdier quotient

$$\mathcal{D}_{\text{sg}}(\Lambda\text{-mod}) = \mathcal{D}^b(\Lambda\text{-mod}) / \mathcal{K}^b(\Lambda\text{-proj}).$$

Recall that  $\Lambda$  is **Gorenstein** if the injective dimensions of  $\Lambda$  as a left  $\Lambda$ -module and as a right  $\Lambda$ -module are finite.

- Gorensteinness is preserved by derived equivalence (A. BELIGIANNIS, 2005).
- Gentle and skewed-gentle algebras are Gorenstein ((CH. GEISS & I. REITEN, 2005) and (X. CHEN & M. LU, 2017)).
- If  $\Lambda$  is Gorenstein, then  $\mathcal{D}_{\text{sg}}(\Lambda\text{-mod}) = \Lambda\text{-}\underline{\text{Gproj}}$  ((R.O. BUCHWEITZ, PREPRINT) and (D. HAPPEL, 1992)).

**Corollary 12.** *If  $\Lambda$  is a derived tame Nakayama algebra, then  $\Lambda$  is Gorenstein, and consequently, if  $\Lambda$  is further a cycle algebra, then  $\mathcal{D}_{\text{sg}}(\Lambda\text{-mod}) = \Lambda\text{-}\underline{\text{Gproj}}$ .*

By using the description of the singularity category of a gentle algebra in (M. KALCK, 2015), we obtain the following result.

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**Corollary 13.** *Let  $\Lambda = \mathbb{k}Q/I$  is a derived tame cycle algebra, and let  $|R_\Lambda|$  the minimal number of relations defining  $I$ . If  $\Lambda$  has infinite global dimension, then there exists an equivalence of triangulated categories*

$$\mathcal{D}_{sg}(\Lambda\text{-mod}) \cong \mathcal{D}^b(\mathbb{k}\text{-mod}) / [|R_\Lambda|],$$

where  $\mathcal{D}^b(\mathbb{k}\text{-mod}) / [|R_\Lambda|]$  denotes the **orbit category** (in the sense of (B. KELLER, 2005)).

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The following result classifies the isomorphism class of versal deformation rings of Gorenstein-projective modules (in the sense of (F. M. BLEHER, V-M, 2012)) over derived tame Nakayama algebras.



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**Corollary 14.** *Let  $\Lambda$  be a derived tame Nakayama algebra, and let  $V$  be in  $\Lambda\text{-Gproj}$ . If  $V$  is indecomposable, then the versal deformation ring  $R(\Lambda, V)$  of  $V$  is universal and isomorphic either to  $\mathbb{k}$  or to  $\mathbb{k}[[t]] / (t^2)$ .*

THANKS FOR YOUR  
ATTENTION!