Derived Tame Nakayama Algebras

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Joint-work with

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- Λ denotes a fixed basic connected finite-dimensional $\mathbb{k}$-algebra.
- Unless explicitly stated, all modules are finitely generated and from the left side.

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- For all projective Λ-modules P, let $\mathbf{r}(P) = (p_1, p_2, \ldots, p_n)$, where $p_1, p_2, \ldots, p_n$ are non-negative integers such that

$$P = \bigoplus_{i=1}^{n} p_i \Lambda\epsilon_i.$$

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$$P = \bigoplus_{i=1}^{n} p_i \Lambda\epsilon_i.$$
- For all complexes of projective Λ-modules $(P^\bullet, \delta^\bullet)$, the **vector rank** $\mathbf{r}^\bullet(P^\bullet)$ is defined as
$$\mathbf{r}^\bullet(P^\bullet) = (\ldots, \mathbf{r}(P^{n-1}), \mathbf{r}(P^n), \mathbf{r}(P^{n+1}), \ldots).$$

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- A **rational family** of bounded minimal complexes over Λ is a bounded complex $(P^\bullet, \delta^\bullet)$ of projective Λ-R-bimodules, where $R = \Bbbk[t, f(t)^{-1}]$ with f a non-zero polynomial, and $\operatorname{Im} \delta^n \subseteq \operatorname{rad} P^{n+1}$.

- For a rational family $(P^\bullet, \delta^\bullet)$, we define the complex $P^\bullet(m,\lambda) = (P^\bullet \otimes_{\mathbb{k}} R/(t-\lambda)^m, \delta^\bullet \otimes 1)$ of projective Λ-modules, where $m \in \mathbb{N}$, $\lambda \in \mathbb{k}$, $f(\lambda) \neq 0$.

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Definition 2 ((V. BEKKERT, YU. DROZD, ARXIV:MATH/0310352)). We say that Λ is **derived tame** if there exists a set $\mathfrak{P}$ of rational families of bounded complexes over Λ such that:

(i) For each bounded vector $\mathbf{v}^\bullet = (\mathbf{v}_i)_{i\in\mathbb{Z}}$ of non-negative integers, the set $\mathfrak{P}(\mathbf{v}^\bullet) = \{P^\bullet \in \mathfrak{P} \mid \mathbf{r}^\bullet(P^\bullet) = \mathbf{v}^\bullet\}$ is finite.

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Theorem 4 ((Ch. Geiss, H. Krause, 2002)). *Derived tameness is preserved by derived equivalence.*

Some Examples of Derived Tame Algebras

- Every derived discrete algebra (in the sense of (D. Vossieck, 2001)) is derived tame.

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Definition 5. Assume that Λ has finite global dimension. The **Euler form** χ_Λ of Λ is defined on the Grothendieck group of Λ by

$$\chi_\Lambda(\mathbf{dim}\, M) = \sum_{i=0}^{\infty}(-1)^i \dim_{\Bbbk} \mathrm{Ext}^i_\Lambda(M, M)$$

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- If Λ is a tree algebra, then Λ is derived tame if and only if χ_Λ is non-negative ((J.A. de la Peña, 1998) & (Th. Brüstle, 2001)).
- If Λ is either gentle or skewed-gentle, then Λ is derived tame (V. Bekkert, H. Merklen, 2003) and (V. Bekkert, E. N. Marcos, H. Merklen, 2003).

Derived Wild Algebras and the Derived Tame-Wild Dichotomy Theorem

Definition 6 ((V. BEKKERT, YU. DROZD, ARXIV:MATH/0310352)). Let $\Sigma = \Bbbk\langle x, y\rangle$. We say that Λ is *derived wild* if there exists a bounded complex $(P^\bullet, \delta^\bullet)$ of projective Λ-Σ-bimodules such that $\operatorname{Im} \delta^n \subseteq \operatorname{rad} P_{n+1}$, and for all Σ-modules L and L' with finite dimension over $\Bbbk$, we have:

(i) $P^\bullet \otimes_\Sigma L \cong P^\bullet \otimes_\Sigma L'$ if and only if $L \cong L'$;

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Question: Which Nakayama algebras Λ are derived tame?

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Theorem 8. *Λ is a Nakayama algebra if and only if $\Lambda = \Bbbk Q/I$, where Q is one of the following quivers:*

$$L_n : \underset{1}{\bullet} \xrightarrow{a_1} \underset{2}{\bullet} \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} \underset{n}{\bullet}$$

$$\begin{tikzcd} C_n : \underset{0}{\bullet} \arrow[r, "a_0"] & \underset{1}{\bullet} \arrow[r, "a_1"] & \cdots \arrow[r, "a_{n-2}"] & \underset{n-1}{\bullet} \arrow[lll, bend left, "a_{n-1}"] \end{tikzcd}$$

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- If $Q = L_n$, then we say that Λ is a **line algebra**.
- If $Q = C_n$, then we say that Λ is a **cycle algebra**.

Theorem 9. (V. BEKKERT, H. GIRALDO, V-M, IN PROGRESS) *Assume that Λ is a Nakayama algebra. Then Λ is derived tame if and only if one of the following conditions holds:*

(i) *Λ is a line algebra whose Euler form is non-negative.*

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Definition 10. (E. ENOCHS, O. JENDA, 1995) A Λ-module V is said to be **Gorenstein-projective** if there exists an acyclic complex of projective Λ-modules

$$P^\bullet : \cdots \to P^{-2} \xrightarrow{\delta^{-2}} P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\delta^0} P^1 \xrightarrow{\delta^1} P^2 \to \cdots$$

such that $\mathrm{Hom}_\Lambda(P^\bullet, \Lambda)$ is also acyclic and $V = \mathrm{coker}\, \delta^0$. We denote by Λ-Gproj the category of Gorenstein-projective Λ-modules that are finitely generated, and by Λ-$\underline{\text{Gproj}}$ its stable category.

Definition 11. The **singularity category** of Λ is defined to be the Verdier quotient

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Corollary 12. *If Λ is a derived tame Nakayama algebra, then Λ is Gorenstein, and consequently, if Λ is further a cycle algebra, then $\mathcal{D}_{sg}(\Lambda\text{-mod}) = \Lambda\text{-}\underline{\mathsf{Gproj}}$.*

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Corollary 13. *Let $\Lambda = \Bbbk Q/I$ is a derived tame cycle algebra, and let $|R_\Lambda|$ the minimal number of relations defining I. If Λ has infinite global dimension, then there exists an equivalence of triangulated categories*

$$\mathcal{D}_{sg}(\Lambda\text{-mod}) \cong \mathcal{D}^b(\Bbbk\text{-mod})/[|R_\Lambda|],$$

where $\mathcal{D}^b(\Bbbk\text{-mod})/[|R_\Lambda|]$ denotes the **orbit category** *(in the sense of* (B. KELLER, 2005)*)*.

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Corollary 14. *Let Λ be a derived tame Nakayama algebra, and let V be in* Λ-Gproj. *If V is indecomposable, then the versal deformation ring $R(\Lambda, V)$ of V is universal and isomorphic either to $\Bbbk$ or to $\Bbbk[[t]]/(t^2)$.*

THANKS FOR YOUR ATTENTION!