Noncommutative Discriminants of Quantum Cluster Algebras

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Joint work with Bach Nguyen and Milen Yakimov

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Maurice Auslander Distinguished Lectures
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Overview

1. Background
   - Discriminants
   - Quantum cluster algebras
2. Quantum cluster algebras at a root of unity
   - Definition
   - Certain subalgebras
   - Main results

Uses of the discriminant of an algebra

The discriminant has been used in:

- determining automorphism groups and solving isomorphism problems for certain PI algebras. [Ceken, Palmieri, Wang, and Zhang]
- solving the Zariski cancellation problem ($A[T] \simeq B[T] \Rightarrow A \simeq B$) in certain cases. [Bell and Zhang]
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Issue

The discriminant can be very difficult to compute directly.

Discriminant $d_n(R/C)$

- Say R is a free, finite rank n module over a subalgebra $C \subset Z(R)$.
- Then the embedding $R \hookrightarrow M_n(C)$ gives a trace map

$$tr\colon R \hookrightarrow M_n(C) \xrightarrow{tr_{M_n(C)}} C.$$

- The **discriminant** of R over C is defined by

$$d_n(R/C) :=_{C^\times} \det\Big(tr(y_i y_j)\Big)$$

where $\{y_1, \cdots, y_n\}$ is a C-basis of R

Quantum cluster algebras [Berenstein and Zelevinsky]

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Ambient field

$\mathrm{Frac}(\mathbb{Z}[x_1, \ldots, x_N])$	$\mathcal{F} = \mathrm{Frac}(\mathcal{T}_q(\Lambda))$ $\mathcal{T}_q(\Lambda) = \mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra with basis X^f, $f \in \mathbb{Z}^N$ and relations $X^f X^g = q^{\frac{\Lambda(f,g)}{2}} X^{f+g}$

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Seeds

$(\widetilde{\mathbf{x}}, \widetilde{B})$	$(M, \widetilde{B})$ with compatibility between M and $\widetilde{B}$ $M : \mathbb{Z}^N \to \mathcal{F}$ such that $\mathcal{T}_q(\Lambda_M) \overset{\phi}{\hookrightarrow} \mathcal{F}$ (1) $M(f) = \phi(f)$ (2) $\mathcal{F} \simeq \mathrm{Frac}(\mathcal{T}_q(\Lambda_m))$

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Mutation $\quad k \in \mathbf{ex} \subset [1, N]$

$\mu_k(\widetilde{\mathbf{x}}, \widetilde{B}) = (\widetilde{\mathbf{x}}', \widetilde{B}')$	$\mu_k(\widetilde{\mathbf{x}}, \widetilde{B}) = (\rho^M_{b^k,s} M E_s, E_s \widetilde{B} F_s)$
x_k replaced by $\dfrac{\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{\lvert b_{ik}\rvert}}{x_k}$	$\mu_k M(e_i) = M(e_i),\ i \neq k$ $\mu_k M(e_k) = M(-e_k + [b^k]_+) + M(-e_k - [b^k]_-)$

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The algebras $\quad \mathbf{inv} \subset [1, N] \backslash \mathbf{ex}$

$\mathcal{A}(\widetilde{B}, \mathbf{inv}) = \mathbb{Z}$-subalgebra	$\mathcal{A}_q(M, \widetilde{B}, \mathbf{inv}) = \mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{F}$
generators: x_j^{-1} for $j \in \mathbf{inv}$ $x_j' \in \widetilde{\mathbf{x}}$ for $(\mathbf{x}', \widetilde{B}') \sim (\mathbf{x}, \widetilde{B})$	generators: $M(e_j)^{-1}$ for $j \in \mathbf{inv}$ $M'(e_j)$ for $(M', \widetilde{B}') \sim (M, \widetilde{B})$

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- Mutation similar to before $\mu_k(M, \widetilde{B}, \Lambda) = (\rho^M_{b^k,s} M E_s, E_s \widetilde{B} F_s, E_s^T \Lambda E_s)$
- $\mathcal{A}_\epsilon(M, \widetilde{B}, \Lambda, \mathbf{inv})$ is the $\mathbb{Z}[\epsilon^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{F}$ generated by $M(e_j)^{-1}, j \in \mathbf{inv}$ and by all $M'(e_j)$ of $(M', \widetilde{B}', \Lambda') \sim (M, \widetilde{B}, \Lambda)$

Some central elements

Lemma

If $(M', \widetilde{B}', \Lambda') \sim (M, \widetilde{B}, \Lambda)$, then $M'(e_j)^\ell \in \mathcal{A}_\epsilon(M, \widetilde{B}, \Lambda)$ is central.

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Proposition

For a quantum seed $(M, \widetilde{B}, \Lambda)$ and ℓ coprime to a finite set of integers dependent on $\widetilde{B}$,

$$(\mu_k M(e_k))^\ell = \frac{\prod_{b_{ik}>0}(M(e_i)^\ell)^{b_{ik}} + \prod_{b_{ik}<0}(M(e_i)^\ell)^{|b_{ik}|}}{M(e_k)^\ell}$$

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Recall

$$x_k' = \frac{\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{|b_{ik}|}}{x_k}$$

A familiar central subalgebra

Theorem (Nguyen–Yakimov–T.)

The (classical) cluster algebra $\mathcal{A}(\widetilde{B}, \mathbf{inv})$ *embeds into the center of* $\mathcal{A}_\epsilon(M, \widetilde{B}, \Lambda, \mathbf{inv})$.

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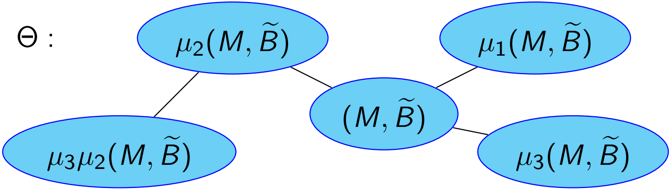
The (classical) cluster algebra $\mathcal{A}(\widetilde{B}, \mathbf{inv})$ embeds into the center of $\mathcal{A}_\epsilon(M, \widetilde{B}, \Lambda, \mathbf{inv})$. Moreover the exchange graphs of $\mathcal{A}(\widetilde{B}, \mathbf{inv})$, $\mathcal{A}_q(M_q, \widetilde{B}, \mathbf{inv})$, and $\mathcal{A}_\epsilon(M, \widetilde{B}, \Lambda, \mathbf{inv})$ all coincide.

$\mathcal{A}_\epsilon(\Theta)$

- Θ is a set of seeds such that every two seeds are connected by sequence of mutations and every nonfrozen direction is mutated at least one time.

Θ :



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- $\mathcal{A}_\epsilon(\Theta) \subset \mathcal{A}_\epsilon(M, \widetilde{B}, \Lambda, \mathbf{inv})$ is the subalgebra that is generated by cluster variables from seeds in Θ and the inverted frozen variables.

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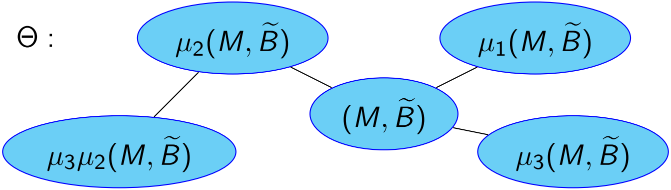

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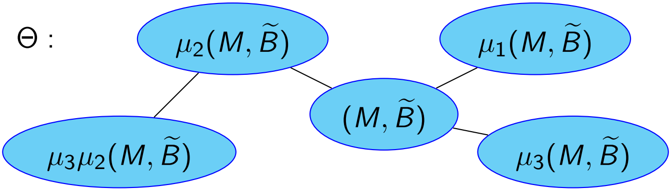
- $\mathcal{C}_\epsilon(\Theta) \subset \mathcal{A}_\epsilon(\Theta)$ is the central subalgebra generated by ℓ^{th} powers of cluster variables from seeds in Θ and the inverted frozen variables.

Θ :



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Theorem (Nguyen–Yakimov–T.)

When $\mathcal{A}_\epsilon(\Theta)$ is free over $\mathcal{C}_\epsilon(\Theta)$, then

$$d_n(\mathcal{A}_\epsilon(\Theta)/\mathcal{C}_\epsilon(\Theta)) = \prod \Big(\textit{noninverted frozen variables of } \mathcal{A}_\epsilon(\Theta) \Big)^{powers}$$

Quantum Schubert cells (or quantum unipotent groups)

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Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. Let $w \in W$ with reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_N}$. $\mathcal{U}_\epsilon^-[w]$ is the subalgebra of $\mathcal{U}_\epsilon[\mathfrak{g}]$ generated by Lusztig root vectors $F_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(F_{i_j})$.

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Theorem (Geiß–Leclerc–Schröer, Goodearl–Yakimov)

$\mathcal{U}_\epsilon^-[w]$ has a canonical cluster algebra structure. The frozen variables are given by generalized minors $\Delta_{\omega_i, w\omega_i}$ for $i \in S(w)$.

Discriminants of quantum Schubert cells

Theorem (Nguyen–Yakimov–T.)

For symmetrizable Kac-Moody algebra $\mathfrak{g}$, $w \in W$, and ϵ a primitive ℓ^{th} root of unity,

$$d_n(\mathcal{U}_\epsilon^-[w]/\mathcal{C}_\epsilon) = \prod_{i \in S(w)} \Delta_{\omega_i, w\omega_i}^{\ell^{N-1}(\ell-1)}$$

Thank you!