Flat modules over noetherian rings with countable spectrum

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Background & Motivation

Classification of flat modules

Even for "simple" rings (e.g. $\mathbb{Z}$), classifying flat modules (using invariants) is a hopeless task. However, we can still try to understand the class via various structure theorems.

E.g.

Theorem (Govorov-Lazard)

Every flat module is the direct limit of free modules of finite rank.

However, this assertion is "clear" in many cases (again, consider $\mathbb{Z}$).

Flat cotorsion pair

The class $\mathcal{F}$ of all flat modules over any ring R together with the class $\mathcal{C}$ of all (Enochs) cotorsion modules constitutes a *cotorsion pair* $(\mathcal{F}, \mathcal{C})$, i.e.

$$\mathcal{F} = {}^{\perp}\mathcal{C} = \{M \in \mathrm{Mod}\text{-}R \mid \mathrm{Ext}^1_R(M, C) = 0 \text{ for all } C \in \mathcal{C}\},$$
$$\mathcal{C} = \mathcal{F}^{\perp} = \{M \in \mathrm{Mod}\text{-}R \mid \mathrm{Ext}^1_R(F, M) = 0 \text{ for all } F \in \mathcal{F}\}.$$

Facts:

- In fact, $\mathrm{Ext}^k_R(F, C) = 0$ for all $F \in \mathcal{F}$, $C \in \mathcal{C}$, $k \geq 1$ (the cotorsion pair is *hereditary*).
- There is a set of modules (not a proper class!) $\mathcal{S}$ such that $\mathcal{C} = \mathcal{S}^{\perp}$; hence $\mathcal{F} = {}^{\perp}(\mathcal{S}^{\perp})$. We say that the cotorsion pair is *generated* by $\mathcal{S}$.

The double-perp class

Theorem (Eklof-Trlifaj)

Let $\mathcal{S} \subset \mathrm{Mod}\text{-}R$ be a set. Then the class ${}^{\perp}(\mathcal{S}^{\perp})$ consists of all direct summands of modules N, such that there is a short exact sequence

$$0 \longrightarrow F \longrightarrow N \longrightarrow S \longrightarrow 0,$$

where F is a free module and S is a transfinite extension of modules from $\mathcal{S}$ (i.e. built using extensions and direct unions).

For example, if $R = \mathbb{Z}$, we obtain the class of all flat modules (= torsion-free groups) via the choice $\mathcal{S} = \{\mathbb{Q}\}$. In that case, S above is just a $\mathbb{Q}$-module.

The double-perp class contd.

More generally,

Theorem (Positselski)

Let R be a commutative noetherian ring of Krull dimension at most 1. Then the class of all flat modules is generated (in the sense above) by the one-element set $\{S^{-1}R\}$, where $S \subset R$ is the multiplicative set consisting of all elements not belonging to any of the minimal prime ideals of R.

Question

Can we produce a similar set of generators for $\mathcal{F}$ for rings of higher dimensions?

Quite flat modules

Quite flat modules

Definition

Let R be a commutative ring and

$$\mathcal{S} = \{S^{-1}R \mid S \subset R \text{ countable multiplicative set}\}.$$

The modules in the class $\mathcal{S}^{\perp}$ are called *almost cotorsion*, while those in ${}^{\perp}(\mathcal{S}^{\perp})$ are called *quite flat*.

Theorem (Positselski-S.)

Let R be a commutative noetherian ring with countable spectrum. Then every flat module is quite flat.

Remark

The projective dimension of quite flat modules cannot exceed 1, so this result cannot hold for “more complicated” rings.

Proof strategy

Main ingredients: Noetherian induction + “Obtainability”

Lemma

Let R be a commutative ring and $S \subset R$ a countable multiplicative set. Then a flat R-module F is quite flat if and only if the $S^{-1}R$-module $S^{-1}F$ is quite flat and the R/sR-module F/sF is quite flat for every $s \in S$.

Noetherian induction

Having this lemma, we can check that a flat module F is quite flat in the following way:

- Let T be the multiplicative set of all elements not belonging to the minimal primes of R. There is a countable multiplicative set S such that $S^{-1}R = T^{-1}R$.
- $S^{-1}R$ is artinian, so $S^{-1}F$ is projective (hence quite flat); therefore, we have to check that F/sF is a quite flat R/sR-module for every $s \in S$. Note that R/sR is noetherian with countable spectrum.
- Thanks to noetherianity of R, we obtain an artinian ring after finitely many passages to a quotient ring as in the previous step; at that point, the flatness turns to projectivity, hence the module will be quite flat and the Lemma inductively shows that F is quite flat.

Obtainability

To prove the Lemma, it is shown that every almost cotorsion R-module C can be "obtained" from almost cotorsion $S^{-1}R$-modules and almost cotorsion R/sR-modules using the following exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_R(S^{-1}R/R, C) \longrightarrow \mathrm{Hom}_R(S^{-1}R, C) \longrightarrow \\ \longrightarrow C \longrightarrow \mathrm{Hom}_{\mathbf{D}(\mathrm{Mod}\text{-}R)}(K^\bullet_{R,S}, C[1]) \longrightarrow 0 \end{aligned}$$

where $S^{-1}R/R$ is a shortcut for the cokernel of the map $R \to S^{-1}R$ and $K^\bullet_{R,S}$ is the complex $\underset{-1}{R} \to \underset{0}{S^{-1}R}$.

Refinements of the results

Further shrinking of the generators

The set

$$\mathcal{S} = \{S^{-1}R \mid S \subset R \text{ countable multiplicative set}\}$$

generating all (quite) flat modules can be replaced by a countable one if the spectrum is countable.

In fact, for finite Krull dimension we can do even better:

Theorem (Positselski-S.)

Let R be a commutative noetherian ring with countable spectrum and Krull dimension $d < \infty$. Then there are countable multiplicative sets $S_1, \dots, S_k$ such that the class of all flats is generated by the set

$$\{S^{-1}R \mid S \text{ is a product of some of } S_1, \dots, S_k\}$$

and $k = \lfloor (d+1)^2/4 \rfloor$.

Example in $k[x, y]$

Example

Let k be a countable field and $R = k[x, y]$. Let

- S = the multiplicative set of all nonzero polynomials *not* containing y,
- T = multiplicative set of all polynomials which *do* contain y.

Then the class of all flats is generated by the set

$$\{S^{-1}R,\ T^{-1}R,\ (ST)^{-1}R\}.$$