Quasi-abelian hearts of twin cotorsion pairs on triangulated categories

Amit Shah

University of Leeds

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Set-up & Motivation

- $\mathcal{C}$ = cluster category (triangulated, Hom-finite, Krull-Schmidt, has Serre duality)

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Aim: to study $\mathrm{mod}\, \Lambda_R$

Buan & Marsh: Hom functors

$$\mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(R,-)} \operatorname{mod} \Lambda_R$$

$$\mathcal{X}_R = \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(R, X) = 0\}$$

Buan & Marsh: Hom functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{Hom}_{\mathcal{C}}(R,-)"] \arrow[d, "Q"'] & \operatorname{mod} \Lambda_R \\
\mathcal{C}/\mathcal{X}_R \arrow[ur, dotted, "\mathrm{Hom}_{\mathcal{C}/\mathcal{X}_R}(R,-)"'] &
\end{tikzcd}$$

where

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- Such a category is *semi-abelian* if $\widetilde{f}$ is *regular*, i.e. simultaneously monic and epic, for all f.

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- The class $\mathcal{R}$ of all regular morphisms admits a calculus of fractions. [Rump]

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\mathcal{C}/\mathcal{X}_R \arrow[rr, "L"'] & & (\mathcal{C}/\mathcal{X}_R)[\mathcal{R}^{-1}] \arrow[u, "\simeq"', "\mathrm{Hom}_{(\mathcal{C}/\mathcal{X}_R)[\mathcal{R}^{-1}]}(R,-)"]
\end{tikzcd}$$

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Let $\mathcal{S}, \mathcal{T}$ be nice subcategories of $\mathcal{C}$.

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A *cotorsion pair* is a pair $(\mathcal{S}, \mathcal{T})$ such that:

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Note: $\mathcal{S} = {}^{\perp_1}\mathcal{T} := \{X \in \mathcal{C} \mid \operatorname{Ext}^1_{\mathcal{C}}(X, \mathcal{T}) = 0\}$, and $\mathcal{T} = \mathcal{S}^{\perp_1}$.

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Example

The pair $(\mathrm{add}\,\Sigma R, \mathcal{X}_R)$ is a cotorsion pair.

Recall $\mathcal{X}_R = \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(R, X) = 0\}$.

Nakaoka: Twin cotorsion pairs

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A *twin cotorsion pair* is a pair $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ such that:

- $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ are each cotorsion pairs

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Example

$((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) = ((\operatorname{add} \Sigma R, \mathcal{X}_R), (\mathcal{X}_R, \mathcal{X}_R^{\perp_1}))$

Example

If $(\mathcal{S}, \mathcal{T})$ is a cotorsion pair, then $((\mathcal{S}, \mathcal{T}), (\mathcal{S}, \mathcal{T}))$ is a *degenerate* twin cotorsion pair.

Nakaoka: the heart

Assume from now that $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is a twin cotorsion pair, and define

$$\mathcal{W} := \mathcal{T} \cap \mathcal{U},$$

$$\mathcal{C}^- := \Sigma^{-1}\mathcal{S} * \mathcal{W},$$

$$\mathcal{C}^+ := \mathcal{W} * \Sigma\mathcal{V}.$$

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The associated *heart* is defined to be

$$\overline{\mathcal{H}} := \mathcal{C}^{-} \cap \mathcal{C}^{+}/\mathcal{W}$$

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and is semi-abelian.

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Recall: Buan-Marsh show $\mathcal{C}/\mathcal{X}_R$ is integral.

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$$\overline{\mathcal{H}} = \mathcal{C}/\mathcal{X}_R.$$

- If $\mathcal{U} \subseteq \mathcal{S} * \mathcal{T}$ or $\mathcal{T} \subseteq \mathcal{U} * \mathcal{V}$, then $\overline{\mathcal{H}}$ is integral.

Main Result

A *quasi-abelian* category is a semi-abelian category in which PBs of cokernels are cokernels and POs of kernels are kernels.

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Theorem (S.)

Let $((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ *be a twin cotorsion pair on a triangulated category* $\mathcal{C}$*. If* $\mathcal{U} \subseteq \mathcal{T}$ *or* $\mathcal{T} \subseteq \mathcal{U}$*, then* $\overline{\mathcal{H}}$ *is quasi-abelian.*

Main Result

A *quasi-abelian* category is a semi-abelian category in which PBs of cokernels are cokernels and POs of kernels are kernels.

Theorem (S.)

Let $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V}))$ be a twin cotorsion pair on a triangulated category $\mathcal{C}$. If $\mathcal{U}\subseteq\mathcal{T}$ or $\mathcal{T}\subseteq\mathcal{U}$, then $\overline{\mathcal{H}}$ is quasi-abelian.

Setting $((\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})) = ((\operatorname{add}\Sigma R,\mathcal{X}_R),(\mathcal{X}_R,\mathcal{X}_R^{\perp_1}))$, we get

Corollary

$\mathcal{C}/\mathcal{X}_R$ is quasi-abelian.

An interesting consequence!

- $\mathcal{C}/\mathcal{X}_R$ is quasi-abelian: a bunch of Auslander-Reiten theory applies! [S.]

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E.g. irreducible maps must be proper epic or proper monic.

- $\mathcal{C}/\mathcal{X}_R$ is Krull-Schmidt: a bunch of Auslander-Reiten theory applies! [Liu]

E.g. the AR theory of $\mathcal{C}$ induces the AR theory of $\mathcal{C}/\mathcal{X}_R$.

Quasi-abelian meets Krull-Schmidt

Theorem (S.)

Let $\mathcal{A}$ be a Krull-Schmidt quasi-abelian category, and $\xi\colon\ X \xrightarrow{f} Y \xrightarrow{g} Z$ an exact sequence in $\mathcal{A}$. The following are equivalent:

(i) *ξ is an Auslander-Reiten sequence;*

(ii) *$\mathrm{End}_{\mathcal{A}}(X)$ is local and g is right almost split;*

(iii) *$\mathrm{End}_{\mathcal{A}}(Z)$ is local and f is left almost split;*

(iv) *f is minimal left almost split; and*

(v) *g is minimal right almost split.*