t-STRUCTURES IN THE BASE OF A DERIVATOR FOR WHICH THE HEART IS A GROTHENDIECK CATEGORY

Manuel Saorín
(joint with Jan Stovicek and Simone Virili)

MAURICE AUSLANDER DISTINGUISHED LECTURES AND INTERNATIONAL CONFERENCE

WOODS HOLE, MASSACHUSETTS (USA)
April 29, 2018

DEFINITION.- **Let $\mathcal{D} = (\mathcal{D}, \Delta, \Sigma)$ be a triangulated category (for us in this talk, always with arbitrary coproducts). A *t-structure* in $\mathcal{D}$ is a pair $(\mathcal{U}, \Sigma\mathcal{V})$ of full subcategories such that:**

a) $(\mathcal{U}, \mathcal{V})$ is an orthogonal pair

b) $\Sigma\mathcal{U} \subseteq \mathcal{U}$ (or $\Sigma^{-1}\mathcal{V} \subseteq \mathcal{V}$)

c) for each $X \in \mathcal{D}$, there is a triangle

$$U \longrightarrow X \longrightarrow V \stackrel{+}{\longrightarrow},$$

with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

In this case $\mathcal{U}$ is called the *aisle*, $\mathcal{V} = \mathcal{U}^{\perp}$ the *co-aisle*, and $\mathcal{H} := \mathcal{U} \cap \Sigma\mathcal{V}$ the *heart* of the t-structure.

PROPOSITION (Beilinson-Bernstein-Deligne).- The heart $\mathcal{H}$ is an abelian category (which in our case is also cocomplete) on which the short exact sequences are the triangles in $\mathcal{D}$ with their three vertices in $\mathcal{H}$.

EXAMPLE OF T-STRUCTURE (Happel-Reiten-Smal∅).- **Let $\mathcal{G}$ be a Grothendieck category and let $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{G}$. Then $(\mathcal{U}_{\mathrm{t}}, \Sigma\mathcal{V}_{\mathrm{t}}) = (\mathcal{U}_{\mathrm{t}}, \Sigma\mathcal{U}_{\mathrm{t}}^{\perp})$ is a t-structure in $\mathcal{D}(\mathcal{G})$, where:**

$$\mathcal{U}_{\mathrm{t}} = \{X \in \mathcal{D}^{\leq 0}(\mathcal{G})\colon\ H^0(X) \in \mathcal{T}\}$$
$$\Sigma\mathcal{V}_{\mathrm{t}} = \{Y \in \mathcal{D}^{\geq -1}(\mathcal{G})\colon\ H^{-1}(Y) \in \mathcal{F}\}.$$

The corresponding heart $\mathcal{H}_{\mathrm{t}} = \mathcal{U}_{\mathrm{t}} \cap \Sigma\mathcal{V}_{\mathrm{t}}$ consists of the complexes M such that $H^{-1}(M) \in \mathcal{F}$, $H^0(M) \in \mathcal{T}$ and $H^i(M) = 0$, for all $i \neq -1, 0$.

REMARK.- If $\mathbf{t} = (\mathcal{G}, 0)$ in the above example, then $(\mathcal{U}_{\mathrm{t}}, \mathcal{V}_{\mathrm{t}}[1]) = (\mathcal{D}^{\leq 0}(\mathcal{G}), \mathcal{D}^{\geq 0}(\mathcal{G}))$. This is the *canonical t-structure*. Its heart $\mathcal{H}$ consists of the complexes whose homology is concentrated in degree 0, so that we clearly have $\mathcal{H} \cong \mathcal{G}$.

DEFINITION.- An abelian category $\mathcal{G}$ is said to be *AB5* or to have the *AB5 property* when it is cocomplete and direct limits are exact in $\mathcal{G}$. If, in addition, $\mathcal{G}$ has a generator, then $\mathcal{G}$ is called a *Grothendieck category.*

ITERATIVE EXAMPLE: If I is any small category and $\mathcal{G}$ is any Grothendieck category, then the category $\mathcal{G}^I = [I, \mathcal{G}]$ of functors $I \longrightarrow \mathcal{G}$ is again a Grothendieck category with all limits and colimits caculated pointwise. In particular, the *category of chain complexes* $\mathcal{C}(\mathcal{G})$ is a Grothendieck category.

DEFINITION (Gabriel-Zisman).- Let $\mathcal{C}$ be a category and $\mathcal{S}$ be a class of morphisms in $\mathcal{C}$. The *localization of $\mathcal{C}$ by $\mathcal{S}$* is a pair $(\mathcal{C}[\mathcal{S}^{-1}], q)$, where:

1. **$\mathcal{C}[\mathcal{S}^{-1}]$ is a category and $q : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ is a functor such that $q(s)$ is an isomorphism, for all $s \in \mathcal{S}$;**
2. **The pair is universal with respect to property 1**

REMARK.- The localization $\mathcal{C}[\mathcal{S}^{-1}]$ may not exist in general, but, when it exists, one has that Ob$(\mathcal{C}[\mathcal{S}^{-1}]) =$ Ob$(\mathcal{C})$ and the functor $q : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$ is the identity on objects.

EXAMPLE.- A (bicomplete) *model category* is a (bicomplete) category $\mathcal{C}$ together with a triple of classes of morphisms $(\mathcal{W}, \mathcal{B}, \mathcal{F})$, called respectively the *weak equivalences*, the *cofribations* and the *fibrations*, satisfying certain axioms that guarantee the existence of functorial factorizations of morphisms $f = \alpha \circ \beta$ and $f = \gamma \circ \delta$, such that $\alpha \in \mathcal{W} \cap \mathcal{F}$, $\beta \in \mathcal{B}$, $\gamma \in \mathcal{F}$ and $\delta \in \mathcal{W} \cap \mathcal{B}$. In such case the localization $\mathcal{C}[\mathcal{W}^{-1}] =:$ Ho$(\mathcal{C})$ exists and is called the *homotopy category* of the model category.

A PARTICULAR EXAMPLE.- **Let** $\mathcal{C} = \mathcal{C}(\mathcal{G})$ **be the category of chain complexes over the Grothendieck category** $\mathcal{G}$ **and consider** $(\mathcal{W}, \mathcal{B}, \mathcal{F})$ **as follows:**

- $\mathcal{W} = qis$ **is the class of quasi-isomorphisms (i.e. morphisms which induce isomorphisms on homology)**

- $\mathcal{B}$ **is the class of monomorphisms**

- $\mathcal{F}$ **the class of epimorphisms with fibrant kernel (a complex** F **is fibrant when the map** $\mathbf{Hom}_{\mathcal{C}(\mathcal{G})}(Y, F) \xrightarrow{u^*} \mathbf{Hom}_{\mathcal{C}(\mathcal{G})}(X, F)$ **is an epimorphism, for each morphism** $u : X \longrightarrow Y$ **in** $\mathcal{W} \cap \mathcal{B}$**).**

The triple $(\mathcal{W}, \mathcal{B}, \mathcal{F})$ **defines a structure of model category on** $\mathcal{C}(\mathcal{G})$ **usually known as the** ***injective model structure*** **on** $\mathcal{C}(\mathcal{G})$**. The homotopy category** $\mathbf{Ho}(\mathcal{C}(\mathcal{G})) = \mathcal{C}(\mathcal{G})[qis^{-1}]$ **is precisely the** ***derived category*** **of** $\mathcal{G}$**.**

THE MOTIVATING QUESTION: When is the heart of a t-structure a Grothendieck category?

TWO PARTIAL RESULTS (Parra-Saorín, 2016-2017):

THEOREM 1.- The heart of the HRS t-structure in $\mathcal{D}(\mathcal{G})$ associated to a torsion pair $\mathbf{t} = (\mathcal{T}, \mathcal{F})$ is a Grothendieck category if, and only if, $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$

THEOREM 2.- If R is a commutative noetherian ring, then almost all compactly generated t-structures in $\mathcal{D}(R-\text{Mod})$ have a heart which is a Grothendieck category.

MAIN TOOL TO PROVE THE AB5 CONDITION: To lift some direct system $(M_i)_{i \in I}$ in the heart (whence in $\mathcal{D}(\mathcal{G})$) to a direct system $(\hat{M}_i)_{i \in I}$ in $\mathcal{C}(\mathcal{G})$ in such a way that $q(\varinjlim_{\mathcal{C}(\mathcal{G})} \hat{M}_i) = \varinjlim_{\mathcal{H}} M_i$, where $q : \mathcal{C}(\mathcal{G}) \longrightarrow \mathcal{D}(\mathcal{G})$ is the canonical functor. Then exactness of direct limits in $\mathcal{H}$ is derived from the corresponding property in (the Grothendieck category) $\mathcal{C}(\mathcal{G})$.

DEFINITION.- Let Cat denote the 2-category of small categories and CAT the 2-category of all categories. A *pre-derivator* is a (strict) 2-functor $\mathbb{D} : Cat^{op} \longrightarrow CAT$. This means:

1. **For each $I \in Cat$, $\mathbb{D}(I)$ is a category;**
2. **For each functor $u : I \longrightarrow J$ in Cat, we have a functor $u^* := \mathbb{D}(u) : \mathbb{D}(J) \longrightarrow \mathbb{D}(I)$;**
3. **If $\alpha : u \Longrightarrow v$ is a natural transformation of functors in Cat, then we get a natural transformation $\alpha^* := \mathbb{D}(\alpha) : u^* \Longrightarrow v^*$ of functors in CAT.**
4. **Some (technical) compatibility conditions between these functors and natural transformations.**

DEFINITION.- A pre-derivator $\mathbb{D} : Cat^{op} \longrightarrow CAT$ is a *derivator* when it satisfies the following axioms:

- (Der.1) If $\coprod_{i \in I} J_i$ is a disjoint union of small categories, then the canonical functor $\mathbb{D}(\coprod_{i \in I} J_i) \longrightarrow \prod_{i \in I} \mathbb{D}(J_i)$ is an equivalence of categories.

- (Der.2) For any $I \in Cat$ and any morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ in $\mathbb{D}(I)$, one has that f is an isomorphism if (and only if) $i^*(f) : i^*(\mathcal{X}) \longrightarrow i^*(\mathcal{Y})$ is an isomorphism in $\mathbb{D}(\mathbf{1})$, for each $i \in I$.

- (Der.3) For each functor $u : I \longrightarrow J$ in Cat, the associated functor $u^* : \mathbb{D}(J) \longrightarrow \mathbb{D}(I)$ has both a left adjoint $u_!$ and a right adjoint u_* (they are called the left and right *homotopy Kan extensions* of u, respectively).

- (Der.4) The homotopy Kan extensions can be computed pointwise

PROPOSITION (Groth).- If a derivator $\mathbb{D} : Cat^{op} \longrightarrow CAT$ is *stable* (whatever it means!), then $\mathbb{D}(I)$ is a triangulated category, for each $I \in Cat$, and $u^* : \mathbb{D}(J) \longrightarrow \mathbb{D}(I)$ is a triangulated functor, for each functor $u : I \longrightarrow J$ in Cat (and hence also $u_!$ and u_* are triangulated) .

DEFINITION.- A *strong derivator* is a derivator $\mathbb{D}$ such that, for each $I \in Cat$, the canonical functor $\mathbb{D}(\mathbf{2} \times I) \longrightarrow \mathbb{D}(I)^{\mathbf{2}}$ is full and essentially surjective, for all $I \in Cat$.

THEOREM (Cisinski).- Let $(\mathcal{C}, \mathcal{W}, \mathcal{B}, \mathcal{F})$ be a cocomplete model category and, for each $I \in Cat$, let $\mathcal{W}_I$ be the the class of all morphisms in $\mathcal{C}^I$ which are pointwise in $\mathcal{W}$. Then the localization $\mathcal{C}^I[\mathcal{W}_I^{-1}]$ exists and the assignment $I \rightsquigarrow \mathbb{D}_{(\mathcal{C},\mathcal{W})}(I) := \mathcal{C}^I[\mathcal{W}_I^{-1}]$ is the definition on objects of a strong derivator, which is stable whenever the initial model category is stable. The base of this derivator is $\mathbb{D}_{(\mathcal{C},\mathcal{W})}(\mathbf{1}) = \mathcal{C}[\mathcal{W}^{-1}] = \mathbf{Ho}(\mathcal{C})$ is the homotopy category of the given model category.

PARTICULAR CASE: Let $\mathcal{G}$ be a Grothendieck category and consider the injective model structure on $\mathcal{C}(\mathcal{G})$. Cisinski's theorem says that, for each $I \in Cat$, the category $\mathcal{C}(\mathcal{G})^I \cong \mathcal{C}(\mathcal{G}^I)$ can be localized with respect to the class $\mathcal{W}_I$ of pointwise quasi-isomorphisms in $\mathcal{C}(\mathcal{G})^I$, which can be identified with the class of quasi-isomorphisms in $\mathcal{C}(\mathcal{G}^I)$. As a consequence, we get a strong stable derivator $\mathbb{D}_{\mathcal{G}} : Cat^{op} \longrightarrow CAT$ such that $\mathbb{D}_{\mathcal{G}}(I) \cong \mathcal{C}(\mathcal{G})^I[\mathcal{W}_I^{-1}] \cong \mathcal{D}(\mathcal{G}^I)$.

DEFINITION.- Let $\mathbb{D} : Cat^{op} \longrightarrow CAT$ be a strong stable derivator. A *t-structure on* $\mathbb{D}$ is a pair $(\mathbb{U}, \Sigma\mathbb{V})$ of subprederivators of $\mathbb{D}$ such that $(\mathbb{U}(I), \Sigma\mathbb{V}(I))$ is a t-structure on $\mathbb{D}(I)$, for each $I \in Cat$. The induced t-structure $(\mathcal{U}, \Sigma\mathcal{V}) := (\mathbb{U}(\mathbf{1}), \Sigma\mathbb{V}(\mathbf{1}))$ in $\mathcal{D} := \mathbb{D}(\mathbf{1})$ will be called the *base of* $(\mathbb{U}, \Sigma\mathbb{V})$.

REMARK.- Note that implicit in the definition above is the fact that if $u : I \longrightarrow J$ is a functor in Cat, then the triangulated functor $u^* : \mathbb{D}(J) \longrightarrow \mathbb{D}(I)$ satisfies that $u^*(\mathbb{U}(J)) \subseteq \mathbb{U}(I)$ and $u^*(\mathbb{V}(J)) \subseteq \mathbb{V}(I)$ (this is not true for the Kan extensions of u!!!). In particular, when taking $I = \mathbf{1}$ and $u = j : \mathbf{1} \longrightarrow J$, where $j \in J$, we get that $j^*(\mathbb{U}(J)) \subseteq \mathbb{U}(\mathbf{1}) = \mathcal{U}$ and $j^*(\mathbb{V}(J)) \subseteq \mathbb{V}(\mathbf{1}) = \mathcal{V}$. Therefore we have inclusions:

$$\mathbb{U}(J) \subseteq \{\mathcal{X} \in \mathbb{D}(J)\text{: } \mathcal{X}_j := j^*(\mathcal{X}) \in \mathcal{U}, \text{ for all } j \in J\} =: \mathcal{U}_J$$

and

$$\mathbb{V}(J) \subseteq \{\mathcal{Y} \in \mathbb{D}(J)\text{: } \mathcal{Y}_j := j^*(\mathcal{Y}) \in \mathcal{V}, \text{ for all } j \in J\} =: \mathcal{V}_J$$

THEOREM 1 (Saorín-Stovicek-Virili 2018).- Let $\mathbb{D} : Cat^{op} \longrightarrow CAT$ a strong stable derivator with base $\mathbb{D}(\mathbf{1}) = \mathcal{D}$. The assignment $(\mathbb{U}, \Sigma\mathbb{V}) \rightsquigarrow (\mathbb{U}(\mathbf{1}), \Sigma\mathbb{V}(\mathbf{1}))$ gives a one-to-one correspondence between

a) t-structures on the derivator $\mathbb{D}$.

b) t-structures on $\mathcal{D}$.

The inverse bijection takes $(\mathcal{U}, \Sigma\mathcal{V}) \rightsquigarrow (\mathbb{U}, \Sigma\mathbb{V})$, where $\mathbb{U}(I) = \mathcal{U}_I$ and $\mathbb{V}(I) = \mathcal{V}_I$, for all $I \in Cat$.

Moreover, if $\mathcal{H}_I = \mathcal{U}_I \cap \Sigma\mathcal{V}_I$ is the heart of $(\mathcal{U}_I, \Sigma\mathcal{V}_I)$, then the canonical functor $\text{dia}_I : \mathbb{D}(I) \longrightarrow \mathbb{D}(\mathbf{1})^I = \mathcal{D}^I$ induces an equivalence of categories $\mathcal{H}_I \xrightarrow{\cong} \mathcal{H}^I$, where $\mathcal{H} = \mathcal{U} \cap \Sigma\mathcal{V}$ is the heart of the base.

DEFINITION.- Let $\mathbb{D} : Cat^{op} \longrightarrow CAT$ a strong stable derivator with base $\mathbb{D}(\mathbf{1}) = \mathcal{D}$. A t-structure $(\mathcal{U}, \Sigma\mathcal{V})$ in $\mathcal{D}$ is said to be *homotopically smashing (wrt $\mathbb{D}$)* when, for each <u>directed</u> small category (equivalent, each directed set) I, the functor Hocolim $: \mathbb{D}(I) \longrightarrow \mathbb{D}(\mathbf{1}) = \mathcal{D}$ satisfies that $\text{Hocolim}(\mathcal{V}_I) \subseteq \mathcal{V}$.

EXAMPLE.- Let $\mathrm{t} = (\mathcal{T}, \mathcal{F})$ be a torsion pair in the Grothendieck category $\mathcal{G}$. The associated HRS t-structure $(\mathcal{U}_{\mathrm{t}}, \Sigma\mathcal{V}_{\mathrm{t}})$ on $\mathcal{D}(\mathcal{G})$ is homotopically smashing (wrt the derivator $\mathbb{D}_{\mathcal{G}}$) if, and only if, $\mathcal{F}$ is closed under taking direct limits in $\mathcal{G}$.

THEOREM 2 (S-S-V 2018).- Let $\mathbb{D} : Cat^{op} \longrightarrow CAT$ be a strong stable derivator with base $\mathbb{D}(\mathbf{1}) = \mathcal{D}$, let $\tau = (\mathcal{U}, \Sigma\mathcal{V})$ be a t-structure on $\mathcal{D}$ and let $\mathcal{H} = \mathcal{U} \cap \Sigma\mathcal{V}$ be its heart. Consider the following conditions:

1. **τ is compactly generated.**
2. **τ is homotopically smashing (wrt $\mathbb{D}$).**
3. **$\mathcal{H}$ is an AB5 abelian category.**

Then the implications $1) \Longrightarrow 2) \Longrightarrow 3)$ hold and $2) \not\Longrightarrow 1)$.

THEOREM 3 (S-S-V 2018).- Let $\mathcal{C}$ be a combinatorial stable model category, let λ be an infinite regular cardinal and let $(\mathcal{U}, \Sigma\mathcal{V})$ be a t-structure in $\mathcal{D} := \mathrm{Ho}(\mathcal{C})$ such that $\mathcal{V}$ is closed under λ-directed homotopy colimits. The heart $\mathcal{H} = \mathcal{U} \cap \Sigma\mathcal{V}$, which is a cocomplete abelian category, has a generator.

REMARK.- For $\mathcal{C}$ as in Theorem 3, the following are examples of t-structures in $\mathcal{D} = \mathrm{Ho}(\mathcal{C})$ satisfying the requirement, for some infinite regular cardinal λ. In particular their heart has a generator:

1. **The t-structures generated by a set of objects.**
2. **The homotopically smashing t-structures (here one can takes $\lambda = \aleph_0$).**

COROLLARY 3 (S-S-V 2018).- Let $\mathcal{C}$ be a combinatorial stable model category and consider its homotopy category $\mathrm{Ho}(\mathcal{C})$ (e.g. any well-generated algebraic or topological triangulated category is of this form). If $\tau = (\mathcal{U}, \Sigma\mathcal{V})$ is a homotopically smashing (e.g. compactly generated) t-structure on $\mathrm{Ho}(\mathcal{C})$, then its heart $\mathcal{H} = \mathcal{U} \cap \Sigma\mathcal{V}$ is a Grothendieck category.