Tensor products on free abelian categories and Nori motives

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Free abelian categories

Freyd showed that, given a skeletally small preadditive category R, for instance a ring, or the category $\mathrm{mod}\text{-}S$ of finitely presented modules over a ring, there is an embedding $R \to \mathrm{Ab}(R)$ of R into an abelian category which has the following universal property.

for every additive functor $M : R \to \mathcal{A}$, where $\mathcal{A}$ is an abelian category, there is a unique-to-natural-equivalence extension of M to an exact functor $\widetilde{M}$ making the following diagram commute.

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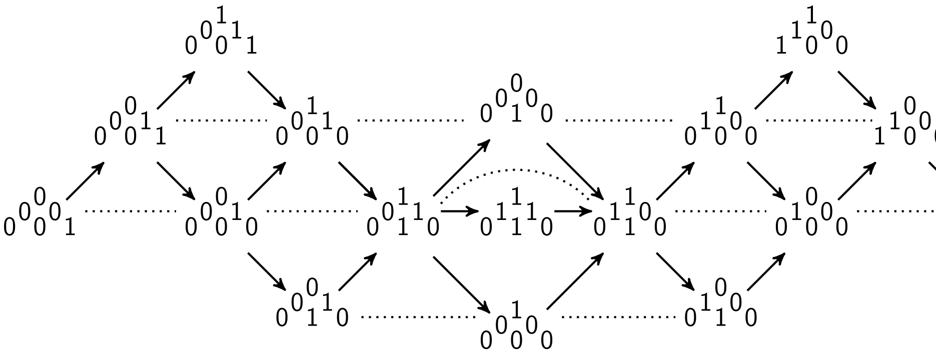
The category $\mathrm{Ab}(R)$ is realised as the category of finitely presented functors on finitely presented left R-modules, or as the category of pp-pairs for left R-modules.

Theorem

For any ring or small preadditive category R, there are natural equivalences $\mathrm{Ab}(R) \simeq (R\text{-}\mathrm{mod}, \mathbf{Ab})^{\mathrm{fp}} \simeq {}_R\mathbb{L}^{\mathrm{eq}+}$. Furthermore, with reference to the diagram above, $\widetilde{M} = M^{\mathrm{eq}+}$, the enrichment of the R-module M by pp-imaginaries.

For Example:

The free abelian category on the quiver A_3 $\bullet \to \bullet \to \bullet$
(rather, on its path algebra, equivalently on the preadditive category freely generated by A_3):



Nori motives

(Grothendieck) The motive of a variety should be its abelian avatar: given a suitable category $\mathcal{V}$ of varieties (or schemes), there should be a functor from $\mathcal{V}$ to its category of motives. That category should be abelian and such that every homology or cohomology theory on $\mathcal{V}$ factors through the functor from $\mathcal{V}$ to its category of motives. So that functor itself should be a kind of universal (co)homology theory for $\mathcal{V}$.
In the case that $\mathcal{V}$ is the category of nonsingular projective varieties over $\mathbb{C}$, there is such a category of motives. But the question of existence for possibly singular, not-necessarily projective varieties - the conjectural category of mixed motives - is open.

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In the 90s Nori described the construction of an abelian category which is a candidate for the category of mixed motives. His idea is to construct from a category of varieties $\mathcal{V}$ a (very large) quiver D such every (co)homology theory on $\mathcal{V}$ gives a representation of D (or D^{op}). A particular representation - singular homology - is then used to construct this category of motives.

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(There is more involved than this, in particular a product structure on D is needed to give a tensor product operation on the category of motives.)

It turns out that Nori's category of motives is a Serre quotient of the free abelian category on D, the quotient being determined by the representation given by singular homology.

In essence this first appeared in a paper of Barbieri-Viale, Caramello and Lafforgue (arXiv:1506:06113), though it is not said this way. In that paper Caramello used the methods of categorical model theory, in particular classifying toposes for regular logic, and showed that Nori's category is the effectivisation of the regular syntactic category for a regular theory associated to Nori's diagram D. This is a much simpler construction than Nori's original one, in particular there is no need to approximate the final result through finite subdiagrams of D or to go *via* coalgebra representations.

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In that paper additivity appears at a relatively late stage of the construction. If we build that in from the beginning then (Barbieri-Viale and Prest, arXiv:1604:00153), we are able to apply the existing model theory of additive structures and, in particular, to realise Nori's category of motives as a localisation of the free abelian category on the preadditive category $\mathbb{Z}\overrightarrow{D}$ generated by Nori's diagram D.
($\overrightarrow{D}$ is the category freely generated by D - so $\mathbb{Z}\overrightarrow{D}$ is essentially the path algebra of D).

The Serre quotient associated to a representation

Theorem

Suppose that M is a representation of the small preadditive category R and let $\widetilde{M}$ be its exact extension to the free abelian category on R. The kernel of $\widetilde{M}$, $\mathcal{S}_M = \{F \in \mathrm{Ab}(R) : \widetilde{M}F = 0\}$, is a Serre subcategory of $\mathrm{Ab}(R)$ and there is a factorisation of $\widetilde{M}$ as a composition of exact functors through the quotient category $\mathcal{A}(M) = \mathrm{Ab}(R)/\mathcal{S}_M$.

$$\begin{tikzcd}
R \arrow[r, "j"] \arrow[dd, "M"'] & \mathrm{Ab}(R) \arrow[ddl, "\widetilde{M}"] \arrow[dr] & \\
& & \mathcal{A}(M) \arrow[dll, "\widehat{M}"] \\
\mathbf{Ab} & &
\end{tikzcd}$$

Nori's diagram

For the vertices, we take triples (X, Y, i) where $X, Y \in \mathcal{V}$, Y is a closed subvariety of X and $i \in \mathbb{Z}$.

The arrows of D are of two kinds:
- for each morphism $f : X \to X'$ of $\mathcal{V}$ we have, for each i, a corresponding arrow $(X, Y, i) \to (X', Y', i)$ provided $fY \subseteq Y'$;
- for each $X, Y, Z \in \mathcal{V}$ with $Y \supseteq Z$ closed subvarieties of X, we add an arrow $(Y, Z, i) \to (X, Y, i-1)$.

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A homology theory H on $\mathcal{V}$ gives a representation of this quiver by sending (X, Y, i) to the relative homology $H_i(X, Y)$. Arrows of the first kind are sent to the obvious maps between relative homology objects; those of the second kind are sent to the connecting maps in the long exact sequence for homology.

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In fact, more is needed. In particular there should be a tensor product structure on motives. This is needed, for example, to express the Künneth formula.
In Barbieri-Viale, Huber and Prest, arXiv:1803.00809, we show how to induce this structure. In particular we show how a tensor product on the category of R-modules induces a tensor product on the free abelian category $\mathrm{Ab}(R)$.

Lifting tensor product from modules to functors on modules

Suppose that $R\text{-mod}$ has a tensor product. Then there is an induced tensor product on the free abelian category $\mathrm{Ab}(R) = (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$, defined as follows.

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Given $A, B \in R$-mod, define $\otimes$ on the corresponding representable functors by $(A, -) \otimes (B, -) = (A \otimes B, -)$.
Given morphisms $f : A \to A'$ and $g : B \to B'$ between finitely presented modules, define $(f, -) \otimes (g, -) = (f \otimes g, -) : (A' \otimes B', -) \to (A \otimes B, -)$.
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The tensor product constructed on $\mathrm{Ab}(R)$ will be required to be right exact, so that forces the rest of the construction.
A typical object of $\mathrm{Ab}(R)$ is the cokernel of a morphism between representables:

$$(B, -) \xrightarrow{(f, -)} (A, -) \xrightarrow{\pi} F_f \to 0$$

for some morphism $f : A \to B$.
Therefore if $C \in R\text{-}\mathrm{mod}$ then the value of $(C, -) \otimes F_f$ is forced by requiring the sequence

$$(C, -) \otimes (B, -) \to (C, -) \otimes (A, -) \xrightarrow{\pi} (C, -) \otimes F_f \to 0$$

to be exact.
That can then be repeated to compute the general case $F_g \otimes F_f$.

Example:

$R = K[\epsilon : \epsilon^2 = 0]$ is commutative so we have the usual $\otimes$ on $R\text{-}\mathrm{mod}$

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where K is the unique simple R-module.

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where K is the unique simple R-module.

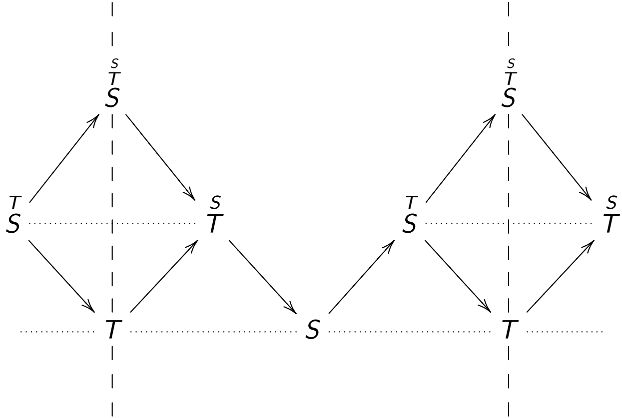
Using this, we get the projective presentations of the two simple functors on $R\text{-mod}$:

$$0 \longrightarrow (K, -) \xrightarrow{(p,-)} (R, -) \xrightarrow{\pi_S} S = F_p \longrightarrow 0$$

$$0 \longrightarrow (K, -) \xrightarrow{(p,-)} (R, -) \xrightarrow{(j,-)} (K, -) \xrightarrow{\pi_T} T = F_j \longrightarrow 0.$$

The category $\mathrm{Ab}(K[\epsilon])$:

$$
\begin{tikzcd}
 & (R,-) \arrow[dr] & & & & (R,-) \arrow[dr] & \\
(K,-) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & (R,-)/\mathrm{soc}(R,-) \arrow[dr] & & (K,-) \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & {} \\
{} \arrow[r, dotted, no head] & T \arrow[ur] \arrow[rr, dotted, no head] & & S \arrow[ur] \arrow[rr, dotted, no head] & & T \arrow[ur] \arrow[r, dotted, no head] & {}
\end{tikzcd}
$$



We compute the values of $\otimes$ on $\mathrm{Ab}(R)$ using the projective presentations

$$(K, -) \xrightarrow{(p,-)} (R, -) \xrightarrow{\pi_S} S \to 0$$

and

$$(R, -) \xrightarrow{(j,-)} (K, -) \xrightarrow{\pi_T} T \to 0$$

of the simple functors S and T.

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of the simple functors S and T.

To compute $S \otimes S$:

$$\begin{tikzcd}
(K \otimes K,-) \arrow[r, "(p\otimes 1_K,-)"] \arrow[d, "(1_K\otimes p,-)"'] & (R \otimes K,-) \arrow[r, "\pi_S\otimes(1_K,-)"] \arrow[d, "(1_R\otimes p,-)"'] & S \otimes (K,-) \arrow[r] \arrow[d, "1_S\otimes(p,-)"'] & 0 \\
(K \otimes R,-) \arrow[r, "(p\otimes 1_R,-)"] \arrow[d, "(1_K,-)\otimes\pi_S"'] & (R \otimes R,-) \arrow[r, "\pi_S\otimes(1_R,-)"] \arrow[d, "(1_R,-)\otimes\pi_S"'] & S \otimes (R,-) \arrow[r] \arrow[d, "1_S\otimes\pi_S"'] & 0 \\
(K,-) \otimes S \arrow[r, "(p,-)\otimes 1_S"] \arrow[d] & (R,-) \otimes S \arrow[r, "\pi_S\otimes 1_S"] \arrow[d] & S \otimes S \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

which simplifies to:

$$
\begin{tikzcd}
(K,-) \arrow[r, "1"] \arrow[d, "1"'] & (K,-) \arrow[r] \arrow[d, "(p,-)"'] & 0 \arrow[r] \arrow[d] & 0 \\
(K,-) \arrow[r, "(p,-)"] \arrow[d] & (R,-) \arrow[r, "\pi_S"] \arrow[d, "\pi_S"'] & S \arrow[r] \arrow[d, "\pi_S \otimes 1_S"] & 0 \\
0 \arrow[r] \arrow[d] & S \arrow[r, "\pi_S \otimes 1_S"] \arrow[d] & S \otimes S \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}
$$

Hence $S \otimes S = S$ and $\pi_S \otimes 1_S = 1_S$.

	S	T	$(K,-)=\overset{T}{S}$	$\overset{S}{T}$	$(R,-)=\overset{S}{\overset{T}{S}}$
S	S	0	0	S	S
T	0	$(K,-)$	$(K,-)$	T	T
$(K,-)$	0	$(K,-)$	$(K,-)$	$(K,-)$	$(K,-)$
$\overset{S}{T}$	S	T	$(K,-)$	$\overset{S}{T}$	$\overset{S}{T}$
$(R,-)$	S	T	$(K,-)$	$\overset{S}{T}$	$(R,-)$

A final remark: this shows how, when we have a tensor product on R-mod, to form the tensor product of pp formulas.