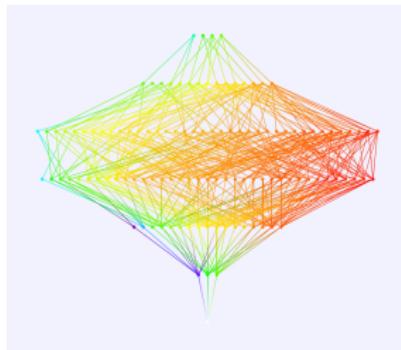


Elementary (super) groups

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Auslander Days 2018
Woods Hole

DETECTION QUESTIONS

Let G be some algebraic object so that

$$\mathbf{Rep} G, \quad H^*(G)$$

make sense.

Question (1)

How to detect that an element $\xi \in H^*(G)$ is nilpotent?

Question (2)

Let $M \in \mathbf{Rep} G$. How to detect projectivity of M ?

Question (3)

$\mathfrak{T}(G)$ - tt - category associated to G ($\mathrm{stmod}\, G, D^b(G), K(\mathrm{Inj} G) \dots$)

$$\mathrm{supp}\, M = \emptyset \Leftrightarrow M \cong 0 \text{ in } \mathfrak{T}(G)$$

- G - finite group, finite group scheme $H^*(G, k)$.
 - G - algebraic group, $H^*(G, A)$
 - G - compact Lie group (p-local compact group)
 - G - Hopf algebra
 - small quantum group (char 0)
 - restricted enveloping algebra of a p-Lie algebra
 - Lie superalgebra
 - Nichols algebra
 - G - finite supergroup scheme

“Other” contexts:

- Stable Homotopy Theory: Devinatz - Hopkins - Smith ('88)
 - Commutative Algebra: $\mathbb{D}^{\text{perf}}(R\text{-mod})$, $\mathbb{D}(R\text{-mod})$, Hopkins ('87), Neeman ('92)
 - Algebraic Geometry: $\mathbb{D}^{\text{perf}}(\text{coh}(X))$, Thomason ('97)

HISTORICAL FRAMEWORK: FINITE GROUPS

Nilpotence in cohomology: D. Quillen, B. Venkov, *Cohomology of finite groups and elementary abelian subgroups*, 1972

Projectivity on elementary abelian subgroups: L. Chouinard, *Projectivity and relative projectivity over group rings*, 1976

Projectivity on shifted cyclic subgroup; finite dimensional modules: E. C. Dade. *Endo-permutation modules over p -groups*, 1978

Dade's lemma for infinite dimensional modules: D.J. Benson, J.F. Carlson, J.Rickard, *Complexity and varieties for infinitely generated modules I, II*, 1995, 1996

G -finite group. $k = \overline{\mathbb{F}}_p$.

$\text{Rep } G$ - abelian category with enough projectives (proj = inj).

- $H^i(G, k) = \text{Ext}_G^i(k, k)$, an abelian group for every i .
- $H^*(G, k) = \text{Ext}_G^*(k, k) = \bigoplus \text{Ext}_G^i(k, k)$ - graded commutative algebra;
 $H^*(G, M) = \text{Ext}_G^*(k, M)$ - module over $H^*(G, k)$ via Yoneda product.

Theorem (Golod ('59), Venkov ('61), Evens('61))

Let G be a finite group. Then $H^*(G, k)$ is a finitely generated k -algebra.
If M is a finite dimensional G -module, then $H^*(G, M)$ is a finite module over $H^*(G, k)$.

$E = (\mathbb{Z}/p)^{\times n}$ - an elementary abelian p -group of rank n .

$$H^*(E, k) = k[Y_1, \dots, Y_n] \otimes \underbrace{\Lambda^*(s_1, \dots, s_n)}_{\text{nilpotents}}, \quad p > 2$$

$$E < G \quad \rightsquigarrow \quad \text{res}_{G,E} : H^*(G, k) \rightarrow H^*(E, k)$$

Theorem (Quillen '71, Quillen-Venkov '72)

A cohomology class $\xi \in H^*(G, k)$ is nilpotent if and only if for every elementary abelian p -subgroup $E < G$,

$$\text{res}_{G,E}(\xi) \in H^*(E, k)$$

is nilpotent.

We say that nilpotence in cohomology *is detected* on elementary abelian p -subgroups.

QUILLEN STRATIFICATION

$$H^*(E, k) = k[Y_1, \dots, Y_n] \otimes \underbrace{\Lambda^*(s_1, \dots, s_n)}_{\text{nilpotents}}.$$

$$|E| = \text{Spec } H^*(E, k) = \text{Spec } k[Y_1, \dots, Y_n] \simeq \mathbb{A}^n$$

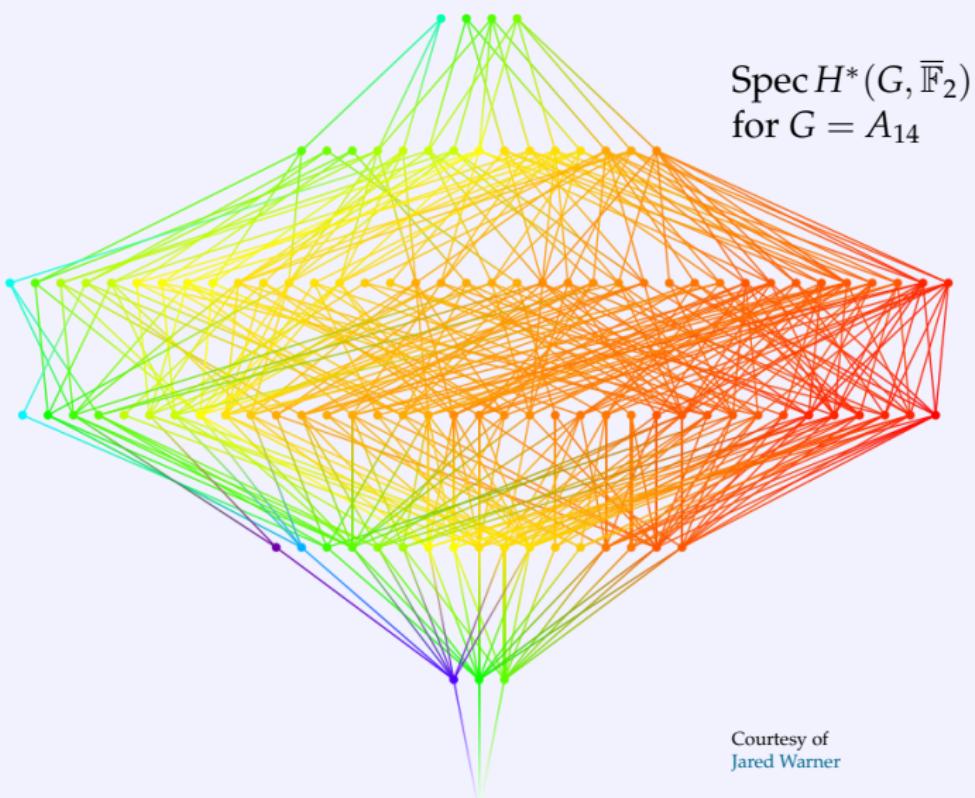
Theorem (Quillen, '71)

$|G| = \text{Spec } H^*(G, k)$ is *stratified* by $|E|$, where $E < G$ runs over all elementary abelian p -subgroups of G .

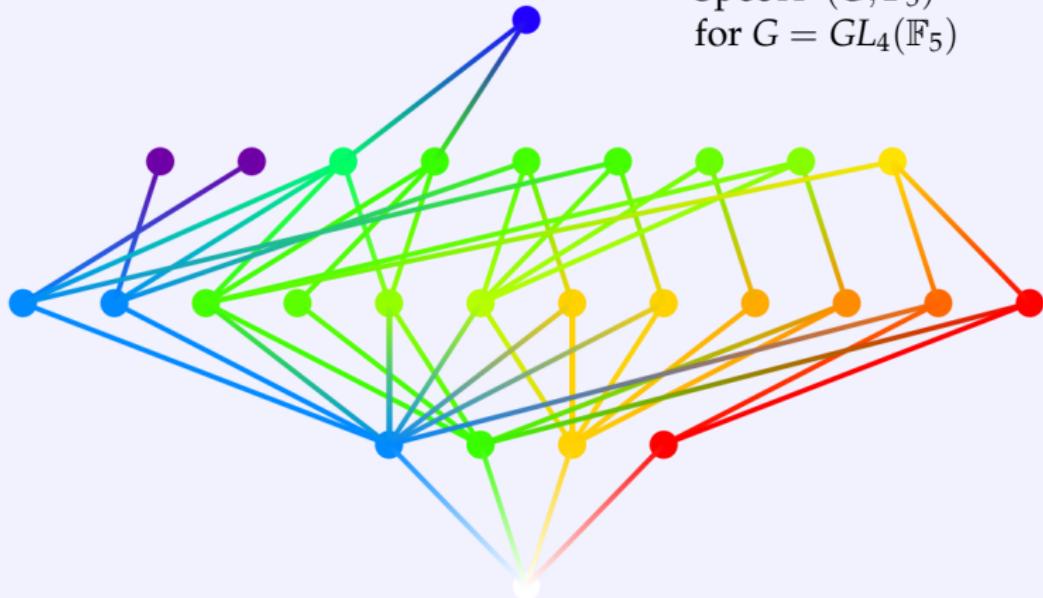
“Weak form” of Quillen stratification:

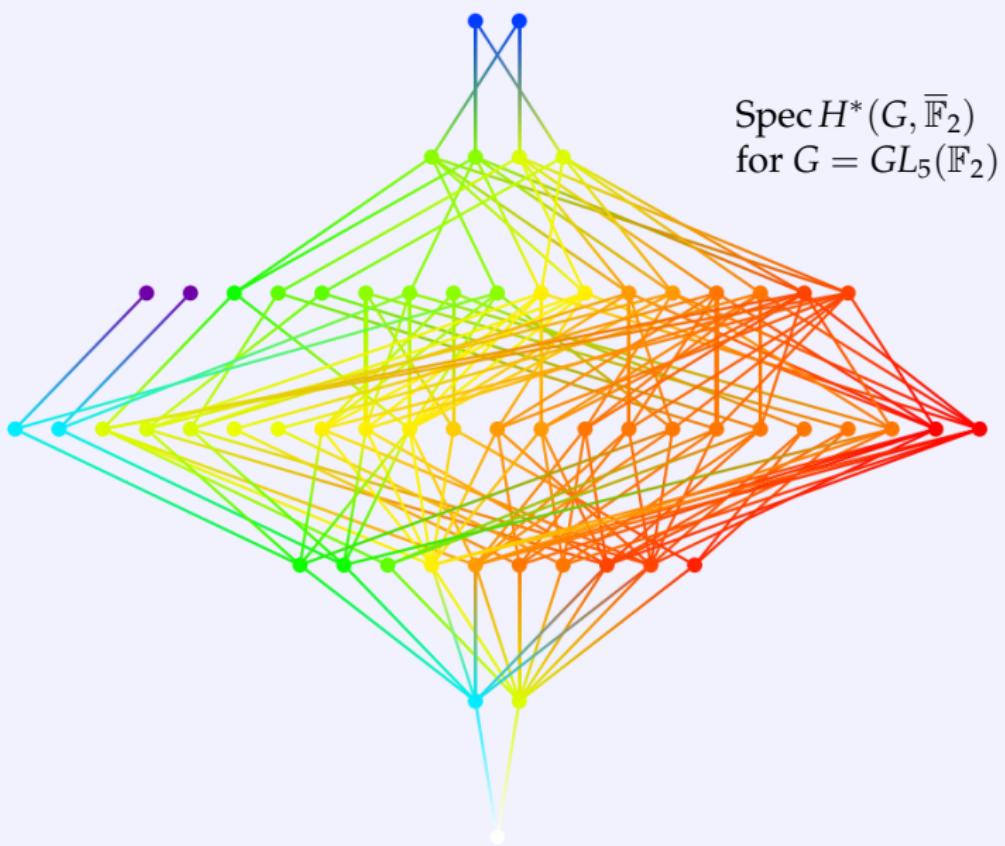
$$|G| = \bigcup_{E < G} \text{res}_{G,E} |E|$$

QUILLEN STRATIFICATION IN GRAPHICS

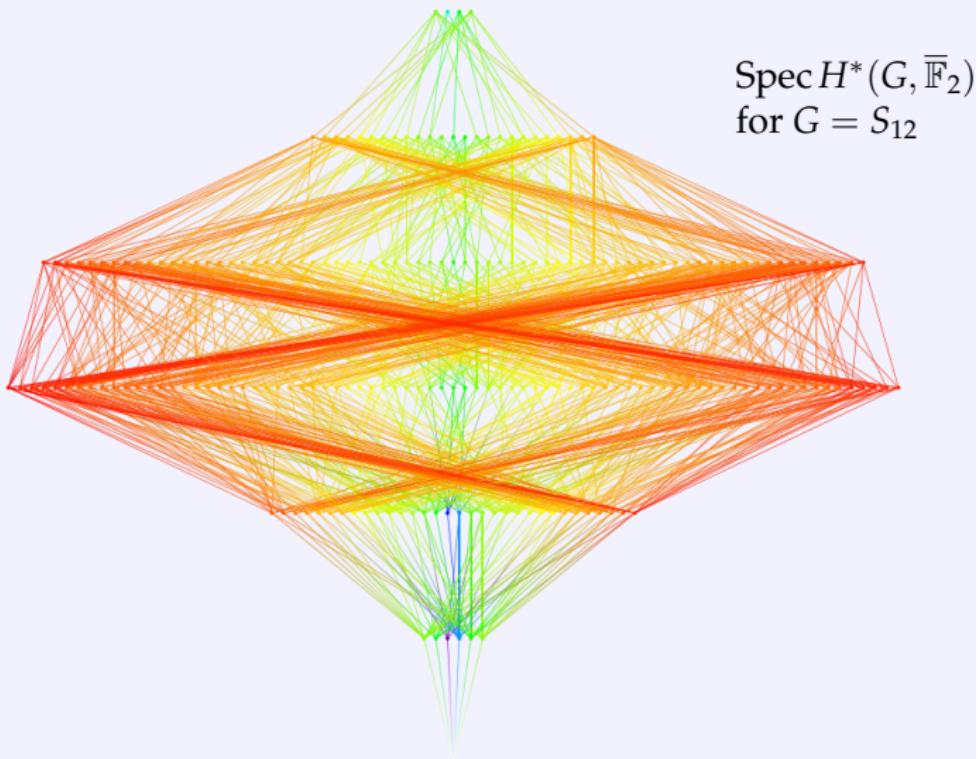


$\text{Spec } H^*(G, \bar{\mathbb{F}}_5)$
for $G = GL_4(\mathbb{F}_5)$





$\text{Spec } H^*(G, \overline{\mathbb{F}}_2)$
for $G = GL_5(\mathbb{F}_2)$



DETECTION FOR MODULES

Theorem (Chouinard '76)

Let G be a finite group, and M be a G -module. Then M is projective if and only for any elementary abelian p -subgroup E of G , $M|_E$ is projective.

“Projectivity is detected on elementary abelian p -subgroups”.

What about elementary abelian p -subgroups?

What about elementary abelian p -subgroups?

Let $E = (\mathbb{Z}/p)^{\times n}$, $(\sigma_1, \sigma_2, \dots, \sigma_n)$ be generators of E . Then

$$kE \simeq \frac{k[\sigma_1, \sigma_2, \dots, \sigma_n]}{(\sigma_i^p - 1)} \simeq \frac{k[x_1, \dots, x_n]}{(x_1^p, \dots, x_n^p)}.$$

where $x_i = \sigma_i - 1$.

$$\lambda = (\lambda_1, \dots, \lambda_n) \in k^n \quad \mapsto \quad X_\lambda = \lambda_1 x_1 + \dots + \lambda_n x_n \in kE.$$

Freshman calculus rule: $X_\lambda^p = 0$, $(X_\lambda + 1)^p = 1$.

Hence, $\langle X_\lambda + 1 \rangle \cong \mathbb{Z}/p$ is a *shifted* cyclic subgroup of kE .

Theorem (Dade'78)

Let E be an elementary abelian p -group, and M be a finite dimensional E -module. Then M is projective if and only if for any $\lambda \in k^n \setminus \{0\}$, $M \downarrow_{\langle X_\lambda + 1 \rangle}$ is projective (free).

APPLICATIONS

- Support varieties for G -modules (Alperin-Evans, Carlson, Avrunin-Scott, ...)
- Classification of thick tensor ideals in $\text{stmod } G$; localizing tensor ideals in $\text{Stmod } G$ (Benson-Carlson-Rickard'97; Benson-Iyengar-Krause'11)
- Computation of Balmer spectrum of $\text{stmod } G$.

FINITE GROUP SCHEMES

An *affine group scheme* over k is a representable functor

$$G : \text{comm } k\text{-alg} \rightarrow \text{groups}$$

R - commutative k -algebra. $\rightsquigarrow G(R) = \text{Hom}_{k\text{-alg}}(k[G], R)$.

$k[G]$ is a commutative Hopf algebra.

An affine group scheme is *finite* if $\dim_k k[G] < \infty$.

$$\left\{ \begin{array}{c} \text{finite group} \\ \text{schemes} \\ G \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional} \\ \text{commutative} \\ \text{Hopf algebras} \\ k[G] \end{array} \right\}$$

G - a finite group scheme.

$kG := k[G]^\vee = \text{Hom}_k(k[G], k)$, the **group algebra** of G , a finite-dimensional *cocommutative* Hopf algebra

$$\left\{ \begin{array}{c} \text{finite group} \\ \text{schemes} \\ G \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional} \\ \text{cocommutative} \\ \text{Hopf algebras} \\ kG \end{array} \right\}$$

$$\boxed{\text{Rep}_k G} \sim \boxed{k[G]\text{-comodules}} \sim \boxed{kG\text{-modules}}$$

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$\boxed{\text{Rep}_k G}$

\sim

$\sim \boxed{kG\text{-modules}}$

Abuse of language: G -modules

$\text{Rep } G = \text{Mod } G$ - abelian category with enough projectives (proj=inj)

$H^*(G, k) = H^*(kG, k)$ - graded commutative algebra.

EXAMPLES

- **Finite groups.** kG is a finite dimensional cocommutative Hopf algebra, generated by group like elements.
- **Restricted Lie algebras.**

Let \mathcal{G} be an algebraic group ($\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n$).

Then $\mathfrak{g} = \mathrm{Lie} \mathcal{G}$ is a *restricted Lie algebra*. It has the p -restriction map (or p^{th} -power map)

$$[p] : \mathfrak{g} \rightarrow \mathfrak{g}$$

a semi-linear map satisfying some natural axioms.

For example, for $\mathfrak{g} = gl_n$, $A^{[p]} = A^p$

$$u(\mathfrak{g}) = U(\mathfrak{g}) / \langle x^p - x^{[p]}, x \in \mathfrak{g} \rangle$$

restricted enveloping algebra (f.d. cocommutative Hopf algebra).

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Representations of \mathfrak{g}	~	$u(\mathfrak{g})$ -modules
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- **Frobenius kernels.** $F : \mathcal{G} \rightarrow \mathcal{G}$ - Frobenius map;

$$\mathcal{G}_{(r)} = \text{Ker}\{F^{(r)} : \mathcal{G} \rightarrow \mathcal{G}\}$$

(connected) finite group scheme.

- **Frobenius kernels of the Additive group \mathbb{G}_a .**

$$\mathbb{G}_a(R) := R^+.$$

$$k[\mathbb{G}_a] = k[T], \Delta(T) = T \otimes 1 + 1 \otimes T.$$

$$F : \mathbb{G}_a \xrightarrow{a \mapsto a^p} \mathbb{G}_a$$

$$\mathbb{G}_{a(1)}(R) = \text{Ker } F(R) = \{a \in R \mid a^p = 0\}$$

$$\mathbb{G}_{a(r)}(R) = \text{Ker } F^{(r)}(R) = \{a \in R \mid a^{p^r} = 0\}$$

$$k[\mathbb{G}_{a(r)}] \cong k[T]/T^{p^r}; \quad \Delta(T) = T \otimes 1 + 1 \otimes T$$

$$k\mathbb{G}_{a(r)} \cong k[s_1, \dots, s_n]/(s_1^{p^r}, \dots, s_n^{p^r})$$

Coproduct in $k\mathbb{G}_{a(r)}$ is given by “Witt polynomials”.

FINITE GENERATION OF COHOMOLOGY

Theorem (Friedlander-Suslin, '97)

Let $A = kG$ be a finite dimensional cocommutative Hopf algebra over a field k . Then $H^*(A, k)$ is a finitely generated k -algebra.

If M is a finite dimensional A -module, then $H^*(A, M)$ is a finite module over $H^*(A, k)$.

Projectivity for finite dimensional modules, restricted Lie algebras. E. Friedlander, B. Parshall, *Support varieties for restricted Lie algebras*, 1986.

Nilpotence and projectivity, finite dimensional modules, connected finite group schemes. A. Suslin, E. Friedlander, C. Bendel, *Support varieties for infinitesimal group schemes*, 1997

Nilpotence and projectivity, infinite dimensional modules, unipotent finite groups schemes. C. Bendel, *Cohomology and projectivity of modules for finite group schemes*, 2001

Projectivity, infinite dimensional modules, infinitesimal finite groups schemes. J. Pevtsova, *Infinite dimensional modules for Frobenius kernels*, 2002

Nilpotence, all finite groups schemes. A. Suslin, *Detection theorem for finite groups schemes.*, 2006

Projectivity, infinite dimensional modules, all finite groups schemes. E. Friedlander, J. Pevtsova, *Π -supports for modules for finite groups schemes*, 2007

D. Benson, S. Iyengar, H. Krause, J. Pevtsova, *Stratification of module categories for finite groups scheme*, 2018

Definition

An *elementary group scheme* is a finite group scheme isomorphic to $\mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^{\times n}$.

The group algebra is commutative and cocommutative; as an algebra it looks like kE for E an elementary abelian p -group. As a coalgebra it is (way) more complicated but still very explicit.

Definition

A **π -point** α of a finite group scheme G defined over field extension K/k is a flat map of algebras

$$K[t]/t^p \xrightarrow{\alpha} KG$$

which factors through some elementary subgroup scheme $\mathcal{E} \subset G_K$.

Theorem (Suslin'06)

Let G be a finite groups scheme. A class $\zeta \in H^*(G, k)$ is nilpotent if and only if

$$\text{res}_{G_K, \mathcal{E}}(\zeta_K) \in H^*(\mathcal{E}, K)$$

is nilpotent for any field extension K/k and any elementary subgroup scheme $\mathcal{E} < G_K$.

Theorem (Benson-Iyengar-Krause-P'18)

Let G be a finite group scheme, and M be a G -module. Then M is projective if and only if for every field extension K/k and any π -point $\alpha : K[t]/t^p \rightarrow KG$, the $K[t]/t^p$ -module $\alpha^*(M_K)$ is projective (free).

Generalization of Dade + Chouinard in two directions: to all finite group schemes (\sim finite dimensional cocommutative Hopf algebras), and to infinite dimensional modules.

Finite generation + detection “ \Rightarrow ” Theory of supports in $\text{Stmod } G$

FINITE SUPERGROUP SCHEMES

$\text{char } k = p > 2, \bar{k} = k$ (perfect is enough)

$\mathbb{Z}/2$ -graded vector spaces, $\mathbb{Z}/2$ -graded Hopf algebras

$$A = A_{\text{ev}} \oplus A_{\text{odd}}$$

Graded commutative: $a \cdot b = (-1)^{|a||b|} b \cdot a$

Graded cocommutative: $T \circ \Delta = \Delta$, where Δ is the coproduct,

$$T : V \otimes W \rightarrow W \otimes V;$$

$$T(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

$$\left\{ \begin{array}{c} \text{finite supergroup} \\ \text{schemes} \\ G \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional} \\ \mathbb{Z}/2\text{-graded cocommutative} \\ \text{Hopf algebras} \\ A = kG \end{array} \right\}$$

EXAMPLES

- Finite group schemes (\sim finite dimensional cocommutative Hopf algebras): $G = G_{ev}$.
- Restricted Lie superalgebras \rightsquigarrow restricted enveloping algebras \rightsquigarrow f.d. graded cocommutative Hopf algebras.

Definition

\mathbb{G}_a^- is a finite supergroup scheme with coordinate algebra
 $\Lambda^*(v) \simeq k[v]/v^2, |v| = 1, \Delta(v) = v \otimes 1 + 1 \otimes v$

- \mathbb{G}_a^- is self-dual with group algebra $k\mathbb{G}_a^- = k[\sigma]/\sigma^2, |\sigma| = 1$.
- Exterior algebras $\Lambda^*(V)$, corresponding to $\mathbb{G}_a^- \times \dots \times \mathbb{G}_a^-$
- Finite dimensional sub Hopf algebras of the mod p Steenrod algebra ([Z-graded](#)).

COHOMOLOGY

$\text{Rep } G = \text{Mod } kG$ – super k -vector spaces with linear kG -action.

Cohomology $H^{*,*}(G, k) = H^{*,*}(kG, k)$

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Cohomology $H^{*,*}(G, k) = H^{*,*}(kG, k)$ - cohomological degree

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$\text{Rep } G = \text{Mod } kG$ – super k -vector spaces with linear kG -action.

Cohomology $H^{*,*}(G, k) = H^{*,*}(kG, k)$ – internal degree

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Cohomology $H^{*,*}(G, k) = H^{*,*}(kG, k)$

Theorem (Drupieski'16)

Let G be a finite supergroup scheme. Then $H^{,*}(G, k)$ is a finitely generated k -algebra.*

For detection, we need “elementary supergroups”.

WITT VECTORS

$W : \text{comm } k\text{-algebras} \rightarrow \text{groups}$

affine group scheme of **additive Witt vectors**.

$$W(R) = \{(a_0, a_1, \dots) \mid a_i \in R\}$$

$$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (S_0(a_0, b_0), S_1(a_0, a_1, b_0, b_1), \dots),$$

S_i - structure polynomials for the additive Witt vectors.

For example, $S_0 = a_0 + b_0$, $S_1 = a_1 + b_1 + \frac{(a_0+b_0)^p - a_0^p - b_0^p}{p}$.

W_m - the group scheme of Witt vectors of length m

$W_{m,n} := W_{m(n)}$ - the n^{th} Frobenius kernel of W_m

- a finite connected commutative unipotent group scheme.

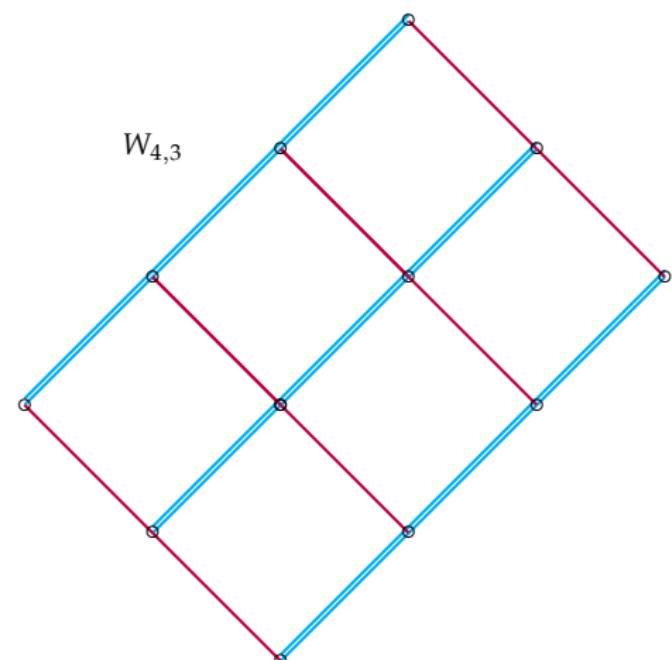
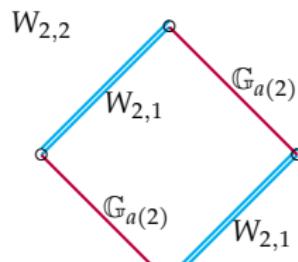
Examples:

- $W_1 \cong \mathbb{G}_a$, $W_{1,n} \cong \mathbb{G}_{a(n)}$
- $W_{m,1} \cong \mathbb{G}_{a(m)}^\vee$ (**Cartier dual**)

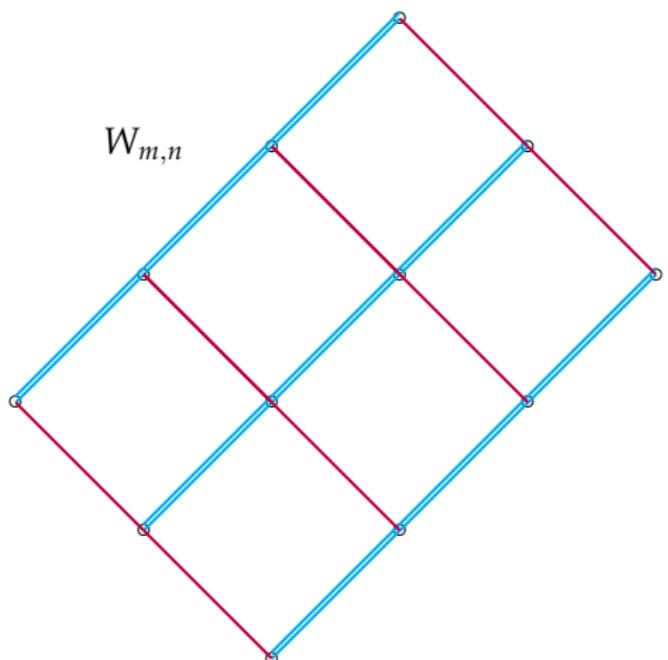
- $W_{2,2}(R) = \{(a_0, a_1) \mid a_0, a_1 \in R\}; \quad kW_{2,2} \cong k[s_0, s_1]/(s_0^{p^2}, s_1^{p^2})$

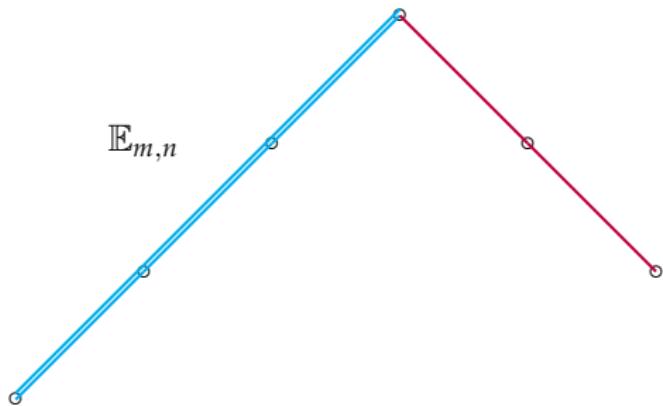
$$\Delta(s_0) = S_0(s_0 \otimes 1, 1 \otimes s_0) = s_0 \otimes 1 + 1 \otimes s_0$$

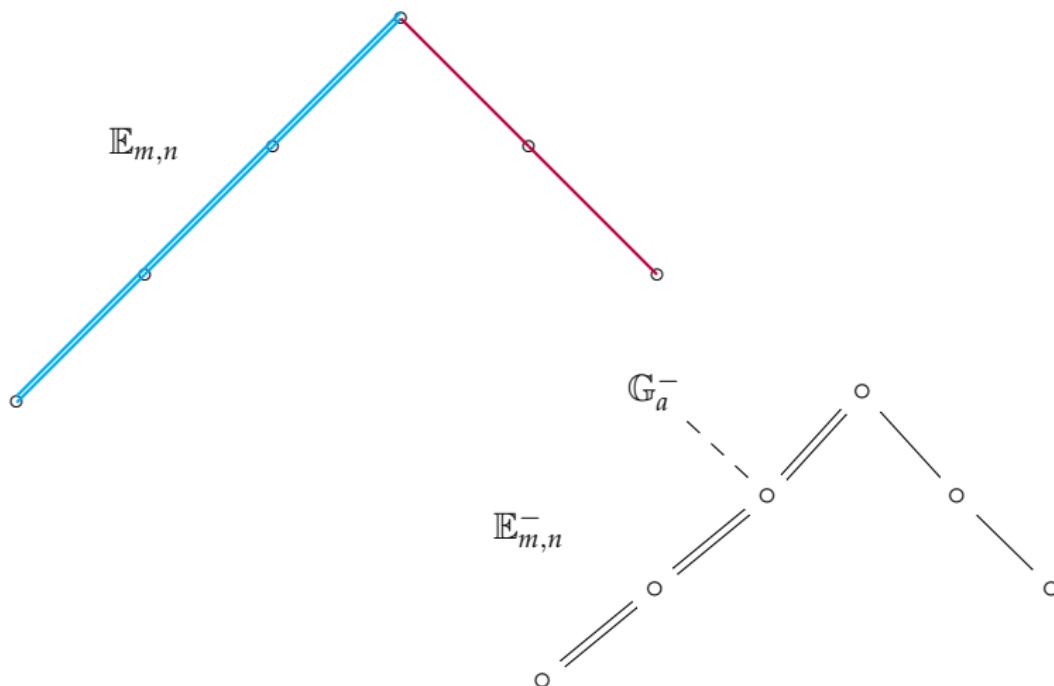
$$\Delta(s_1) = S_1(s_0 \otimes 1, s_1 \otimes 1, 1 \otimes s_0, 1 \otimes s_1) = s_1 \otimes 1 + 1 \otimes s_1 + \frac{(s_0 \otimes 1 + 1 \otimes s_0)^p - (s_0 \otimes 1)^p - (1 \otimes s_0)^p}{p}$$



The simple quotients are $\mathbb{G}_{a(1)}$.







WITT ELEMENTARY SUPERGROUP SCHEMES

(Super) technical part: Witt elementary supergroup schemes

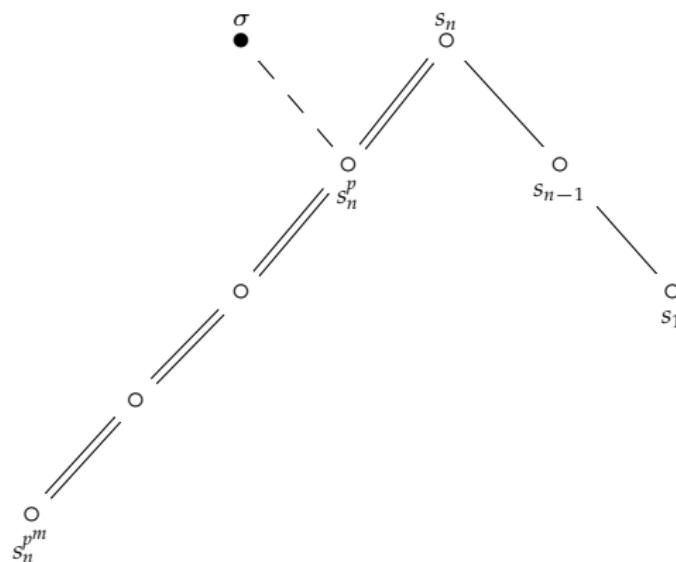
$$k\mathbb{E}_{m,n}^- = \frac{k[s_1, \dots, s_{n-1}, s_n, \sigma]}{(s_1^p, \dots, s_{n-1}^p, s_n^{p^m}, \sigma^2 - s_n^p)}$$

s_1, \dots, s_n are even; σ is odd.

$$\Delta(s_i) = S_{i-1}(s_1 \otimes 1, \dots, s_i \otimes 1, 1 \otimes s_1, \dots, 1 \otimes s_i) \quad (i \geq 1)$$

$$\Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma$$

where the S_i are the structure polynomials for the Witt vectors.

$E_{m,n}^-$ 

Definition

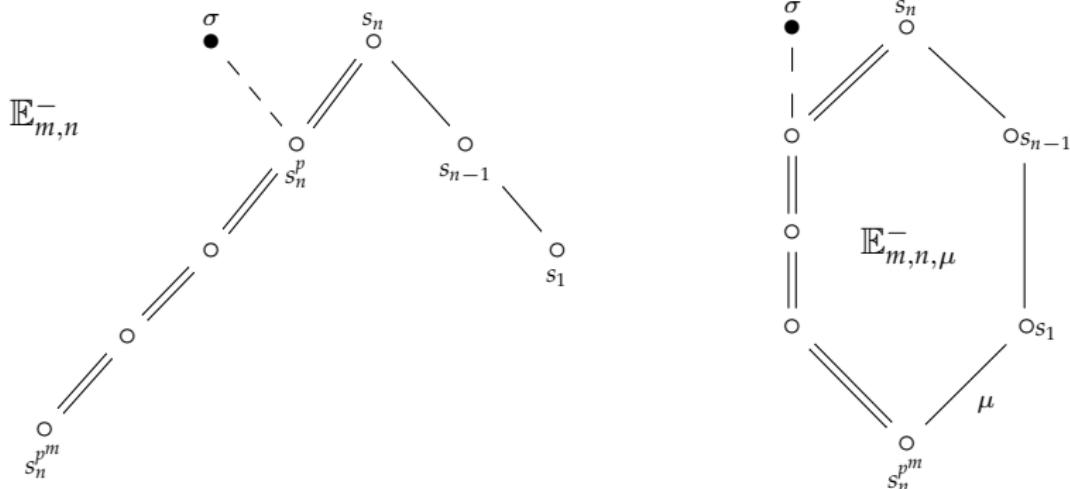
A finite supergroup scheme is **elementary** if it's isomorphic to a quotient of $\mathbb{E}_{m,n}^- \times (\mathbb{Z}/p)^s$.

Remark: These quotients can be explicitly classified using the theory of Diedonné modules.

Theorem (Classification)

An elementary supergroup scheme is isomorphic to one of the following:

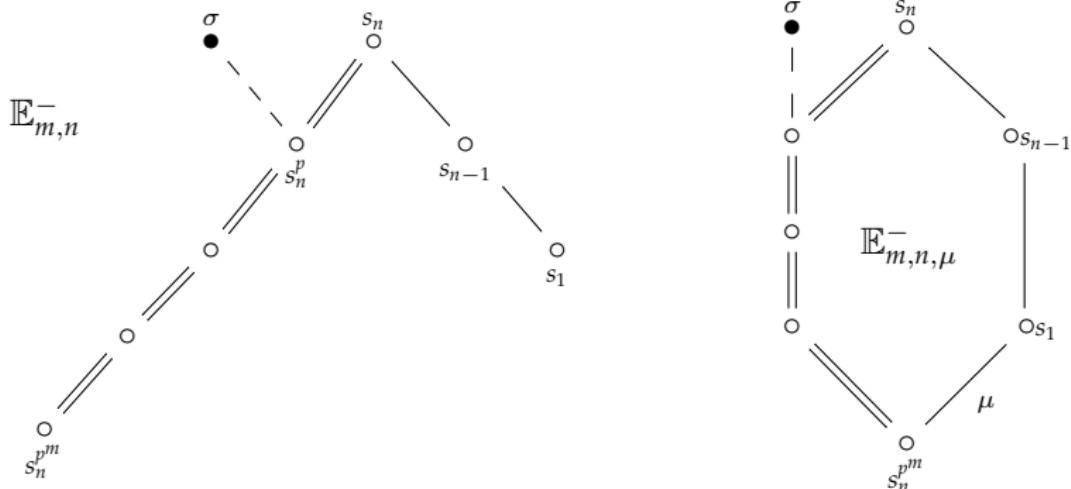
- (i) $\mathbb{G}_{a(n)} \times (\mathbb{Z}/p)^s$,
- (ii) $\mathbb{G}_{a(n)} \times \mathbb{G}_a^- \times (\mathbb{Z}/p)^s$,
- (iii) $\mathbb{E}_{m,n}^- \times (\mathbb{Z}/p)^s$,
- (iv) $\mathbb{E}_{m,n,\mu}^- \times (\mathbb{Z}/p)^s$.



$$k\mathbb{E}_{m,n,\mu}^- = \frac{k[s_1, \dots, s_{n-1}, s_n, \sigma]}{(s_1^p, \dots, s_{n-1}^p, s_n^{p^{m+1}}, \sigma^2 - s_n^p)}$$

$$\Delta(s_i) = S_i(\mu s_n^{p^m} \otimes 1, s_1 \otimes 1, \dots, s_i \otimes 1, 1 \otimes \mu s_n^{p^m}, 1 \otimes s_1, \dots, 1 \otimes s_i)$$

$$\Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma$$



$$kE_{m,n,\mu}^- = \frac{k[s_1, \dots, s_{n-1}, s_n, \sigma]}{(s_1^p, \dots, s_{n-1}^p, s_n^{p^{m+1}}, \sigma^2 - s_n^p)}$$

$$\begin{aligned}\Delta(s_i) &= S_i(\mu s_n^{p^m} \otimes 1, s_1 \otimes 1, \dots, s_i \otimes 1, 1 \otimes \mu s_n^{p^m}, 1 \otimes s_1, \dots, 1 \otimes s_i) \\ \Delta(\sigma) &= \sigma \otimes 1 + 1 \otimes \sigma\end{aligned}$$

DETECTION THEOREM

Theorem (Benson-Iyengar-Krause-P'18)

Suppose that G is a finite unipotent supergroup scheme. Then

- (i) Nilpotence of elements of $H^{*,*}(G, k)$ and
- (ii) Projectivity of G -modules

are detected upon restriction to sub supergroup schemes isomorphic to a quotient of some $\mathbb{E}_{m,n}^- \times (\mathbb{Z}/p)^s$ (after field extension).

THANK YOU