Finite generation of the cohomology rings of some pointed Hopf algebras

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joint work with Xingting Wang and Sarah Witherspoon

Maurice Auslander Distinguished Lectures and International Conference
April 25 – 30, 2018

Setting & Motivation

Let $\mathbf{k}$ be a field and $\mathcal{H}$ be a finite-dimensional Hopf algebra over $\mathbf{k}$.
The **cohomology** of $\mathcal{H}$ is $\mathrm{H}^*(\mathcal{H}, \mathbf{k}) := \bigoplus_{n \geq 0} \mathrm{Ext}^n_{\mathcal{H}}(\mathbf{k}, \mathbf{k})$.

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GOAL: Study the finite generation of $\mathrm{H}^*(\mathcal{H}, \mathbf{k})$, for some pointed Hopf algebras.

Finite generation of cohomology ring

F.g. Cohomology Conjecture

Applications:
Quillen's stratification theorem,
modular representation theory,
support variety theory,
algebraic geometry,
commutative algebra,
some homological conjectures

Partial Results:
finite group algebras over pos. char.,
finite group schemes over pos. char.,
Lusztig's small quantum group over $\mathbb{C}$,
Drinfeld double of Frob. kernels of finite alg. groups,
certain pointed Hopf algebras

Remarks:

- $\mathrm{H}^*(\mathcal{H}, \mathbf{k})$ is a graded-commutative ring.
- $\mathrm{H}^*(\mathcal{H}, \mathbf{k})$ is a finitely generated $\mathbf{k}$-algebra
  $\iff \mathrm{H}^*(\mathcal{H}, \mathbf{k})$ is left (or right) Noetherian
  $\iff \mathrm{H}^{ev}(\mathcal{H}, \mathbf{k})$ is Noetherian and $\mathrm{H}^*(\mathcal{H}, \mathbf{k})$ is a f.g. module over $\mathrm{H}^{ev}(\mathcal{H}, \mathbf{k})$.

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Preliminary Ingredients

Definition

A **Hopf algebra** $\mathcal{H}$ over a field **k** is a **k**-vector space which is an algebra (m, u) ♡ a coalgebra (Δ, ε) ♡ together with an antipode map $S : \mathcal{H} \to \mathcal{H}$.

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Example

group algebra $\mathbf{k}G$, polynomial rings $\mathbf{k}[x_1, x_2, \ldots, x_n]$, universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, etc.

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$$\mathcal{H}_0 = \mathbf{k}C_q = \langle g \rangle, \quad q \text{ is divisible by } p \text{ (more general)},$$
$$\mathrm{gr}\mathcal{H} \cong \mathcal{B}(V)\#\mathbf{k}C_q, \quad \text{where } V = \mathbf{k}x \oplus \mathbf{k}y \text{ is } \mathbf{k}C_q\text{-module.}$$

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- $\mathcal{B}(V)$ is a rank two Nichols algebra of Jordan type over C_q.

$$\mathcal{B}(V) = \mathbf{k}\langle x, y\rangle/(x^p,\ y^p,\ yx - xy - \frac{1}{2}x^2).$$

with action ${}^g x = x$ and ${}^g y = x + y$.

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$$\begin{array}{c} \mathcal{B}(V) \\ \text{Hopf alg in } \mathcal{YD}_{C_q}^{C_q} \end{array} \xrightarrow{\text{bosonization}} \boxed{\mathrm{gr}\mathcal{H} \cong \mathcal{B}(V)\#\mathbf{k}C_q} \xrightarrow{\text{lifting}} \begin{array}{c} \text{pointed Hopf} \\ \text{algebras } \mathcal{H} \end{array}$$

Today's Object: two Hopf algebras

Let $\mathbf{k} = \overline{\mathbf{k}}$ with $\mathrm{char}(\mathbf{k}) = p > 2$ and $w = g - 1$. Consider the following Hopf algebras over $\mathbf{k}$:

1. The p^2q-dim bosonization $\mathrm{gr}\mathcal{H} \cong \mathcal{B}(V)\#\mathbf{k}C_q$ is isomorphic to $\mathbf{k}\langle w, x, y\rangle$ subject to

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2. The 27-dim liftings in $p = q = 3$ are $\mathcal{H} = H(\epsilon, \mu, \tau) \cong \mathbf{k}\langle w, x, y\rangle$ subject to

$$w^3 = 0,\ x^3 = \epsilon x,\ y^3 = -\epsilon y^2 - (\mu\epsilon - \tau - \mu^2)y,$$
$$yw - wy = wx + x - (\mu - \epsilon)(w^2 + w),\ xw - wx = \epsilon(w^2 + w),$$
$$yx - xy = -x^2 + (\mu + \epsilon)x + \epsilon y - \tau(w^2 - w),$$

with $\epsilon \in \{0, 1\}$ and $\tau, \mu \in \mathbf{k}$.

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Main Results (N-Wang-Witherspoon '17)

The cohomology rings of $\mathcal{B}(V)\#\mathbf{k}C_q$ and of $H(\epsilon, \mu, \tau)$ are finitely generated.

Strategy: May spectral sequence & permanent cocycles

- Take $\mathcal{H}$ as $\mathcal{B}(V)\#\mathbf{k}C_q$ or $H(\epsilon, \mu, \tau)$ $(p = q = 3)$.
- Assign lexicographic order on monomials in w, x, y with $w < x < y$.

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- $\mathrm{H}^*(\mathrm{gr}\mathcal{H}, \mathbf{k}) = \bigwedge\left(\mathbf{k}^3\right) \otimes \mathbf{k}\left[\xi_w, \xi_x, \xi_y\right], \ \deg(\xi_i) = 2.$

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- (May '66) May spectral sequence

$$E_1^{*,*}\cong\mathrm{H}^*(\mathrm{gr}\mathcal{H},\mathbf{k})\Longrightarrow E_\infty^{*,*}\cong\mathrm{gr}\,\mathrm{H}^*(\mathcal{H},\mathbf{k}).$$

with respect to the cup product.

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Lemma (Friedlander-Suslin '97)

If ξ_w, ξ_x, ξ_y are permanent cocyles (meaning they survive at E_∞-page), then $\mathrm{gr}\,\mathrm{H}^*(\mathcal{H}, \mathbf{k})$ and $\mathrm{H}^*(\mathcal{H}, \mathbf{k})$ are noetherian over $\mathbf{k}[\xi_w, \xi_x, \xi_y]$. Consequently, $\mathrm{H}^*(\mathcal{H}, \mathbf{k})$ is finitely generated as a $\mathbf{k}$-algebra.

$\Longrightarrow$ Need to find such permanent cocycles!

Tool 1: Twisted tensor product resolution (Shepler-Witherspoon '16)

- Let A and B be associative **k**-algebras.
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- Twisted tensor product resolution: (Shepler-Witherspoon '16) introduced some compatibility conditions that are sufficient for constructing a projective resolution $Y_\bullet = \mathrm{Tot}(P_\bullet(M) \otimes P_\bullet(N))$ of $M \otimes N$ as a module over $A \otimes_\tau B$.

Twisted tensor product resolutions over $\mathcal{B}(V)\#\mathbf{k}C_q$

- $\mathcal{H} = \mathcal{B}(V)\#\mathbf{k}C_q \cong (A \otimes_\tau B) \otimes_\mu C.$
- $A = \mathbf{k}[x]/(x^p), \quad B = \mathbf{k}[y]/(y^p), \quad C = \mathbf{k}[w]/(w^q).$

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- $(\mathcal{K}_\bullet, d)$ is a resolution of $\mathbf{k}$ over $A \otimes_\tau B$.
- $\mathrm{H}^*(\mathcal{K}_\bullet) = \mathrm{H}^*(A \otimes B, \mathbf{k}) \cong \mathrm{H}^*(A, \mathbf{k}) \otimes \mathrm{H}^*(B, \mathbf{k})$.

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- $\mathcal{K}_\bullet$ is $C_q = \langle g \rangle$-equivariant.
- $P_\bullet^{\mathbf{k}C_q}(\mathbf{k}) : \ \cdots \mathbf{k}C_q \xrightarrow{(g-1)\cdot} \mathbf{k}C_q \xrightarrow{(\sum_{s=0}^{q-1} g^s)\cdot} \mathbf{k}C_q \xrightarrow{(g-1)\cdot} \mathbf{k}C_q \xrightarrow{\varepsilon} \mathbf{k} \longrightarrow 0.$
- Twisted tensor resolution $\mathcal{Y}_\bullet := \mathrm{Tot}(\mathcal{K}_\bullet \otimes P_\bullet^{\mathbf{k}C_q}(\mathbf{k}))$ with twisted chain map $\mu_\bullet : \mathbf{k}C_q \otimes K_\bullet \to K_\bullet \otimes \mathbf{k}C_q$ given by the C_q-action.

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Twisted tensor product resolutions over $\mathcal{B}(V)\#\mathbf{k}C_q$

- $\mathcal{K}_n = \bigoplus_{i+j=n} P_i^A(\mathbf{k}) \otimes P_j^B(\mathbf{k}) \cong \bigoplus_{i+j=n} (A \otimes_\tau B)\,\phi_{ij}$.
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- The differentials d are defined recursively, with a simultaneous recursive definition of a contracting homotopy s:

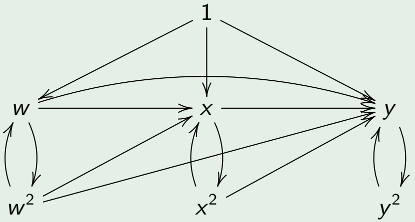
$$\cdots \underset{s_2}{\overset{d_3}{\rightleftarrows}} \mathcal{H} \otimes \mathbf{k}C_2 \underset{s_1}{\overset{d_2}{\rightleftarrows}} \mathcal{H} \otimes \mathbf{k}C_1 \underset{s_0}{\overset{d_1}{\rightleftarrows}} \mathcal{H} \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} \mathbf{k} \longrightarrow 0.$$

Anick resolution for some liftings $\mathcal{H} = H(\epsilon, \mu, \tau)$

Example ($\mathcal{H} = H(\epsilon, \mu, \tau) \cong \mathbf{k}\langle w, x, y\rangle$ subject to)

$$w^3 = 0,\ x^3 = \epsilon x,\ y^3 = -\epsilon y^2 - (\mu\epsilon - \tau - \mu^2)y,$$
$$yw - wy = wx + x - (\mu - \epsilon)(w^2 + w),\ xw - wx = \epsilon(w^2 + w),$$
$$yx - xy = -x^2 + (\mu + \epsilon)x + \epsilon y - \tau(w^2 - w),$$

with $\epsilon \in \{0, 1\}$ and $\tau, \mu \in \mathbf{k}$.



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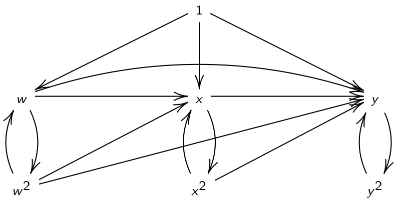

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$$C_3 = \{w^{3+1}, x^{3+1}, y^{3+1}, xw^3, yw^3, yx^3, x^3w, y^3w, y^3x, yxw\}.$$

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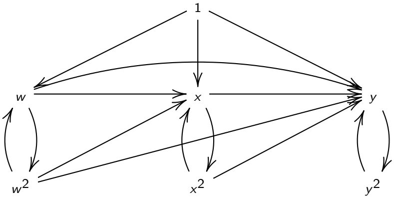

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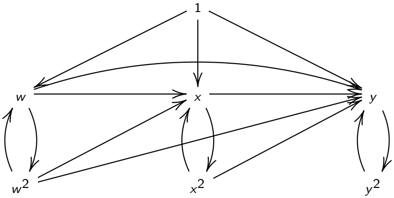

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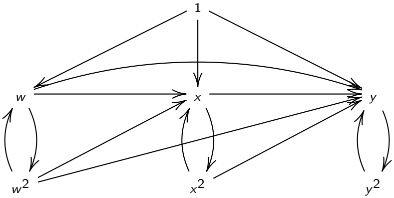

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\begin{aligned}
d_2(1 \otimes w^3) &= w^2 \otimes w, \\
d_2(1 \otimes x^3) &= x^2 \otimes x - \epsilon \otimes x, \\
d_2(1 \otimes y^3) &= y^2 \otimes y + \epsilon y \otimes y + (\mu\epsilon - \tau - \mu^2) \otimes y, \\
d_2(1 \otimes xw) &= x \otimes w - w \otimes x - \epsilon w \otimes w - \epsilon \otimes w, \\
d_2(1 \otimes yw) &= y \otimes w - w \otimes y - w \otimes x - 1 \otimes x + (\mu - \epsilon)w \otimes w + (\mu - \epsilon) \otimes w, \\
d_2(1 \otimes yx) &= y \otimes x - x \otimes y + x \otimes x - (\mu + \epsilon) \otimes x - \epsilon \otimes y + \tau w \otimes w - \tau \otimes w.
\end{aligned}
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$$
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d_3(1 \otimes w^4) &= w \otimes w^3, \quad d_3(1 \otimes x^4) = x \otimes x^3, \quad d_3(1 \otimes y^4) = y \otimes y^3, \\
d_3(1 \otimes xw^3) &= x \otimes w^3 - w^2 \otimes xw, \\
d_3(1 \otimes x^3 w) &= x^2 \otimes xw + w \otimes x^3 + \epsilon wx \otimes xw + \epsilon x \otimes xw + \epsilon w \otimes xw, \\
d_3(1 \otimes yw^3) &= y \otimes w^3 - w^2 \otimes yw + w^2 \otimes xw + w \otimes xw, \\
d_3(1 \otimes yxw) &= y \otimes xw - x \otimes yw + w \otimes yx + \epsilon w \otimes yw + x \otimes xw + (\mu + \epsilon)w \otimes xw, \\
d_3(1 \otimes y^3 w) &= y^2 \otimes yw + w \otimes y^3 + wy \otimes yx + wx \otimes yx + (\epsilon - \mu)wy \otimes yw \\
&\quad + (\mu - \epsilon)wx \otimes yw - \tau w^2 \otimes yw + y \otimes yx - (\epsilon + \mu)y \otimes yw \\
&\quad + \tau w^2 \otimes xw + x \otimes yx + (\mu - \epsilon)x \otimes yw + (\mu^2 - \epsilon\mu)w \otimes yw + \tau w \otimes xw, \\
d_3(1 \otimes yx^3) &= y \otimes x^3 - x^2 \otimes yx + \tau wx \otimes xw + \epsilon x \otimes yx - \tau x \otimes xw + \epsilon\tau w \otimes xw, \\
d_3(1 \otimes y^3 x) &= y^2 \otimes yx + x \otimes y^3 - xy \otimes yx - \tau wx \otimes yw - \tau wy \otimes yw \\
&\quad + \tau w^2 \otimes yx + \tau wx \otimes xw + \epsilon\tau w^2 \otimes yw + (\epsilon\tau + \mu\tau)w^2 \otimes xw \\
&\quad + \mu y \otimes yx + \tau y \otimes yw - \mu x \otimes yx + \tau x \otimes xw \\
&\quad + \tau w \otimes yx + (\epsilon\tau + \mu\tau)w \otimes yw + \epsilon\tau w \otimes xw.
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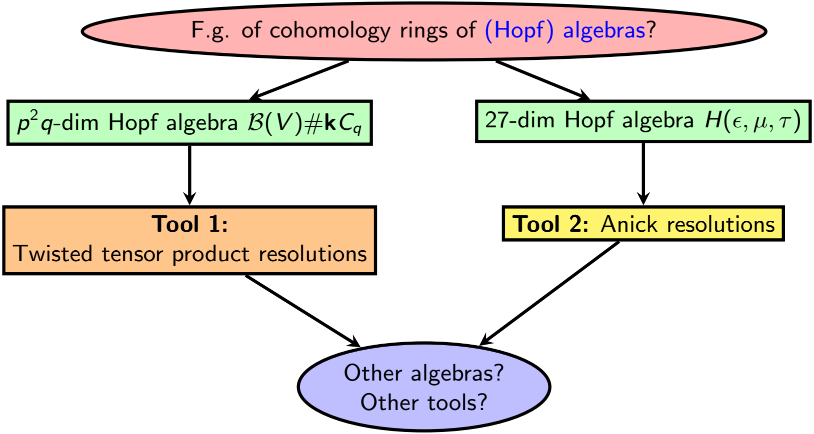
F.g. of cohomology rings of (Hopf) algebras?
p^2q-dim Hopf algebra $\mathcal{B}(V)\#\mathbf{k}C_q$
27-dim Hopf algebra $H(\epsilon, \mu, \tau)$
Tool 1:
Twisted tensor product resolutions
Tool 2: Anick resolutions
Other algebras?
Other tools?

References

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Thank You! ☺