Approximable Triangulated Categories

Amnon Neeman

Australian National University

amnon.neeman@anu.edu.au

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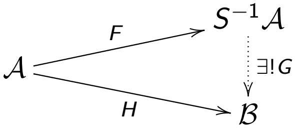
Overview

Let $\mathcal{A}$ be a category, and let S be a class of morphisms in $\mathcal{A}$. There exists a functor $F : \mathcal{A} \longrightarrow S^{-1}\mathcal{A}$ so that

- The functor F takes every morphism in S to an isomorphism.
- If $H : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor taking every morphism in S to an isomorphism, then there exists a unique functor $G : S^{-1}\mathcal{A} \longrightarrow \mathcal{B}$ rendering commutative the triangle



We call this construction *formally inverting the morphisms in S.*

Reminder of the derived categories $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category. The derived category $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ is as follows:

- Objects: cochain complexes of objects in $\mathcal{A}$, that is

$$\cdots \longrightarrow A^{-2} \longrightarrow A^{-1} \longrightarrow A^{0} \longrightarrow A^{1} \longrightarrow A^{2} \longrightarrow \cdots$$

  where the composites $A^i \longrightarrow A^{i+1} \longrightarrow A^{i+2}$ all vanish. The subscript $\mathfrak{C}$ and superscript $\mathfrak{C}'$ stand for conditions.

- Morphisms: cochain maps are examples, that is

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow & A^{0} & \longrightarrow & A^{1} & \longrightarrow & A^{2} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & B^{-2} & \longrightarrow & B^{-1} & \longrightarrow & B^{0} & \longrightarrow & B^{1} & \longrightarrow & B^{2} & \longrightarrow & \cdots \end{array}$$

  but we formally invert the cohomology isomorphisms.

Example

1. If R is a ring, $\mathbf{D}(R)$ will be our shorthand for $\mathbf{D}(R\text{–}\mathrm{Mod})$; the objects are all cochain complexes of R-modules, no conditions.

Let X be a scheme

2. $\mathbf{D_{qc}}(X)$ will be our shorthand for $\mathbf{D_{qc}}(\mathcal{O}_X\text{–}\mathrm{Mod})$. The objects are the complexes of $\mathcal{O}_X$–modules, and the only condition is that the cohomology must be quasicoherent.
3. The objects of $\mathbf{D}^{\mathrm{perf}}(X)$ are the perfect complexes. A complex is *perfect* if it is locally isomorphic to a bounded complex of vector bundles.
4. Assume X is noetherian. The objects of $\mathbf{D}^b_{\mathbf{coh}}(X)$ are the complexes with coherent cohomology which vanishes in all but finitely many degrees.

Definition (formal definition of triangulated categories)

The additive category $\mathcal{T}$ has a *triangulated structure* if:

1. It has an invertible additive endofunctor $[1]:\mathcal{T}\longrightarrow\mathcal{T}$, taking the object X and the morphism f in $\mathcal{T}$ to $X[1]$ and $f[1]$, respectively.
2. We are given a collection of *exact triangles*, meaning diagrams in $\mathcal{T}$ of the form $X\stackrel{f}{\longrightarrow}Y\stackrel{g}{\longrightarrow}Z\stackrel{h}{\longrightarrow}X[1]$.

This data must satisfy the following axioms

[TR1] Any isomorph of an exact triangle is an exact triangle. For any object $X\in\mathcal{T}$ the diagram $0\longrightarrow X\stackrel{\mathrm{id}}{\longrightarrow}X\longrightarrow 0$ is an exact triangle. Any morphism $f:X\longrightarrow Y$ may be completed to an exact triangle $X\stackrel{f}{\longrightarrow}Y\stackrel{g}{\longrightarrow}Z\stackrel{h}{\longrightarrow}X[1]$.

[TR2] Any rotation of an exact triangle is exact. That is: $X\stackrel{f}{\longrightarrow}Y\stackrel{g}{\longrightarrow}Z\stackrel{h}{\longrightarrow}X[1]$ is an exact triangle if and only if $Y\stackrel{-g}{\longrightarrow}Z\stackrel{-h}{\longrightarrow}X[1]\stackrel{-f[1]}{\longrightarrow}Y[1]$ is.

Definition (definition of triangulated categories—continued)

[TR3+4] Given a commutative diagram, where the rows are exact triangles,

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1]
\end{array}$$

we may complete it to a commutative diagram (also known as a morphism of triangles)

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1]
\end{array}$$

Definition (definition of triangulated categories—continued)

[TR3+4] (continued): Moreover: we can do it in such a way that

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -g & 0 \\ v & f' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -h & 0 \\ w & g' \end{pmatrix}} X[1] \oplus Z' \xrightarrow{\begin{pmatrix} -f[1] & 0 \\ u[1] & h' \end{pmatrix}} Y[1] \oplus X'[1]$$

is an exact triangle.

Example (back to $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$)

We have asserted that the category $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ is triangulated.
The endofunctor $[1] : \mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A}) \longrightarrow \mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$: It takes the cochain complex A^*, i.e.

$$\cdots \longrightarrow A^{-2} \xrightarrow{\partial^{-2}} A^{-1} \xrightarrow{\partial^{-1}} A^{0} \xrightarrow{\partial^{0}} A^{1} \xrightarrow{\partial^{1}} A^{2} \longrightarrow \cdots$$

to the cochain complex $\big(A[1]\big)^*$ below:

$$\cdots \longrightarrow A^{-1} \xrightarrow{-\partial^{-1}} A^{0} \xrightarrow{-\partial^{0}} A^{1} \xrightarrow{-\partial^{1}} A^{2} \xrightarrow{-\partial^{2}} A^{3} \longrightarrow \cdots$$

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

If $f^* : A^* \longrightarrow B^*$ is a cochain map

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A^{-2} & \xrightarrow{\partial_A^{-2}} & A^{-1} & \xrightarrow{\partial_A^{-1}} & A^{0} & \xrightarrow{\partial_A^{0}} & A^{1} & \xrightarrow{\partial_A^{1}} & A^{2} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle f^{-2}} & & \Big\downarrow{\scriptstyle f^{-1}} & & \Big\downarrow{\scriptstyle f^{0}} & & \Big\downarrow{\scriptstyle f^{1}} & & \Big\downarrow{\scriptstyle f^{2}} & & \\
\cdots & \longrightarrow & B^{-2} & \xrightarrow[\partial_B^{-2}]{} & B^{-1} & \xrightarrow[\partial_B^{-1}]{} & B^{0} & \xrightarrow[\partial_B^{0}]{} & B^{1} & \xrightarrow[\partial_B^{1}]{} & B^{2} & \longrightarrow & \cdots
\end{array}
$$

then $\big(f[1]\big)^*$ is the cochain map

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A^{-1} & \xrightarrow{-\partial_A^{-1}} & A^{0} & \xrightarrow{-\partial_A^{0}} & A^{1} & \xrightarrow{-\partial_A^{1}} & A^{2} & \xrightarrow{-\partial_A^{2}} & A^{3} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle f^{-1}} & & \Big\downarrow{\scriptstyle f^{0}} & & \Big\downarrow{\scriptstyle f^{1}} & & \Big\downarrow{\scriptstyle f^{2}} & & \Big\downarrow{\scriptstyle f^{3}} & & \\
\cdots & \longrightarrow & B^{-1} & \xrightarrow[-\partial_B^{-1}]{} & B^{0} & \xrightarrow[-\partial_B^{0}]{} & B^{1} & \xrightarrow[-\partial_B^{1}]{} & B^{2} & \xrightarrow[-\partial_B^{2}]{} & B^{3} & \longrightarrow & \cdots
\end{array}
$$

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

The exact triangles: Suppose we are given a commutative diagram in $\mathcal{A}$, where the rows are objects of $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & X^{-2} & \longrightarrow & X^{-1} & \longrightarrow & X^{0} & \longrightarrow & X^{1} & \longrightarrow & X^{2} & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & Y^{-2} & \longrightarrow & Y^{-1} & \longrightarrow & Y^{0} & \longrightarrow & Y^{1} & \longrightarrow & Y^{2} & \longrightarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & Z^{-2} & \longrightarrow & Z^{-1} & \longrightarrow & Z^{0} & \longrightarrow & Z^{1} & \longrightarrow & Z^{2} & \longrightarrow \cdots
\end{array}$$

We may view the above as morphisms $X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^*$ in the category $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$.

Assume further that, for each $i \in \mathbb{Z}$, the sequence $X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i$ is split exact. Choose, for each $i \in \mathbb{Z}$, a splitting $\theta^i : Z^i \longrightarrow Y^i$ of the map $g^i : Y^i \longrightarrow Z^i$.

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

Now for each i we have the diagram

$$\begin{array}{ccccc}
Z^i & \xrightarrow{\ \theta^i\ } & Y^i & \xrightarrow{\ g^i\ } & Z^i \\
\Big\downarrow{\scriptstyle \partial^i_Z} & & \Big\downarrow{\scriptstyle \partial^i_Y} & & \Big\downarrow{\scriptstyle \partial^i_Z} \\
Z^{i+1} & \xrightarrow{\ \theta^{i+1}\ } & Y^{i+1} & \xrightarrow{\ g^{i+1}\ } & Z^{i+1}
\end{array}$$

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

Now for each i we have the diagram

$$\begin{array}{ccccc}
Z^i & \xrightarrow{\theta^i} & Y^i & & \\
\Big\downarrow{\scriptstyle \partial^i_Z} & & \Big\downarrow{\scriptstyle \partial^i_Y} & & \\
Z^{i+1} & \xrightarrow{\theta^{i+1}} & Y^{i+1} & \xrightarrow{g^{i+1}} & Z^{i+1}
\end{array}$$

Example (back to $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$, continued)

Thus the difference $\theta^{i+1}\partial_Z^i - \partial_Y^i\theta^i$ is annihilated by the map $g^{i+1} : Y^{i+1} \longrightarrow Z^{i+1}$, hence must factor uniquely as $Z^i \xrightarrow{h^i} X^{i+1} \xrightarrow{f^{i+1}} Y^{i+1}$. Form the diagram

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & Z^{-2} & \xrightarrow{\partial_Z^{-2}} & Z^{-1} & \xrightarrow{\partial_Z^{-1}} & Z^{0} & \xrightarrow{\partial_Z^{-0}} & Z^{1} & \xrightarrow{\partial_Z^{1}} & Z^{2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle h^{-2}} & & \downarrow{\scriptstyle h^{-1}} & & \downarrow{\scriptstyle h^{0}} & & \downarrow{\scriptstyle h^{1}} & & \downarrow{\scriptstyle h^{2}} & & \\
\cdots & \longrightarrow & X^{-1} & \xrightarrow[-\partial_X^{-1}]{} & X^{0} & \xrightarrow[-\partial_X^{0}]{} & X^{1} & \xrightarrow[-\partial_X^{1}]{} & X^{2} & \xrightarrow[-\partial_X^{2}]{} & X^{3} & \longrightarrow & \cdots
\end{array}
$$

Example (back to $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, continued)

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & Z^{-2} & \xrightarrow{\partial_Z^{-2}} & Z^{-1} & \xrightarrow{\partial_Z^{-1}} & Z^{0} & \xrightarrow{\partial_Z^{-0}} & Z^{1} & \xrightarrow{\partial_Z^{1}} & Z^{2} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle h^{-2}} & & \Big\downarrow{\scriptstyle h^{-1}} & & \Big\downarrow{\scriptstyle h^{0}} & & \Big\downarrow{\scriptstyle h^{1}} & & \Big\downarrow{\scriptstyle h^{2}} & & \\
\cdots & \longrightarrow & X^{-1} & \xrightarrow[-\partial_X^{-1}]{} & X^{0} & \xrightarrow[-\partial_X^{0}]{} & X^{1} & \xrightarrow[-\partial_X^{1}]{} & X^{2} & \xrightarrow[-\partial_X^{2}]{} & X^{3} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle f^{-1}} & & \Big\downarrow{\scriptstyle f^{0}} & & \Big\downarrow{\scriptstyle f^{1}} & & \Big\downarrow{\scriptstyle f^{2}} & & \Big\downarrow{\scriptstyle f^{3}} & & \\
\cdots & \longrightarrow & Y^{-1} & \xrightarrow[-\partial_Y^{-1}]{} & Y^{0} & \xrightarrow[-\partial_Y^{0}]{} & Y^{1} & \xrightarrow[-\partial_Y^{1}]{} & Y^{2} & \xrightarrow[-\partial_Y^{2}]{} & Y^{3} & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccc}
Z^{i} & \xrightarrow[\partial_Z^{i}]{} & Z^{i+1} \\
\Big\downarrow{\scriptstyle h^{i}} & & \Big\downarrow{\scriptstyle h^{i+1}} \\
X^{i+1} & \xrightarrow[-\partial_X^{i+1}]{} & X^{i+2} \\
 & & \Big\downarrow{\scriptstyle f^{i+2}} \\
 & & Y^{i+2}
\end{array}$$

Example (back to $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, continued)

This is a cochain map. We have constructed in the category $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ a diagram $X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \xrightarrow{h^*} X^*[1]$. We declare

- The exact triangles in $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ are all the isomorphs, in $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$, of diagrams that come from our construction.

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & X^{i-2} & \xrightarrow{\partial_X^{i-2}} & X^{i-1} & \xrightarrow{\partial_X^{i-1}} & X^{i} & \xrightarrow{\partial_X^{i}} & X^{i+1} & \xrightarrow{\partial_X^{i+1}} & X^{i+2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \partial_X^{i-1}} & & \| & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & X^{i} & = & X^{i} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

Call this map $\rho_i^* : X^* \longrightarrow C_i^*$. Forming the direct sum over i produces

$$X^* \longrightarrow \bigoplus_{i\in\mathbb{Z}} C_i^*$$

and we call this map $\rho^* : X^* \longrightarrow C^*$.

Given any cochain map $f^* : X^* \longrightarrow Y^*$ we may factor it as

$$X^* \xrightarrow{\begin{pmatrix} \rho^* \\ f^* \end{pmatrix}} C^* \oplus Y^* \xrightarrow{\pi} Y^*$$

Conventions

- If $\mathcal{T}$ is a triangulated category and $n \in \mathbb{Z}$ is an integer, then $[n]$ will be our shorthand for the endofunctor $[1]^n : \mathcal{T} \longrightarrow \mathcal{T}$.
- We will lazily abbreviate "exact triangle" to just "triangle".
- A full subcategory $\mathcal{S} \subset \mathcal{T}$ is called *triangulated* if $0 \in \mathcal{S}$, if $\mathcal{S}[1] = \mathcal{S}$, and if, whenever $X, Y \in \mathcal{S}$ and there exists in $\mathcal{T}$ a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$, we must also have $Z \in \mathcal{S}$.
- The subcategory $\mathcal{S}$ is *thick* if it is triangulated, as well as closed in $\mathcal{T}$ under direct summands.
- Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{A}$ be an abelian category. A functor $H : \mathcal{T} \longrightarrow \mathcal{A}$ is *homological* if it takes triangles to long exact sequences.

Source of homological functors

Let $\mathcal{T}$ be a triangulated category, and assume $A \in \mathcal{T}$ is any object. Then the functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, A)$ are homological.

Remark

For the next two theorems we will suppose R is a noetherian ring, and X is a scheme proper over R. It is an old theorem that, for any pair of objects $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and $B \in \mathbf{D}^b_{\mathbf{coh}}(X)$, we have

1. $\mathrm{Hom}\big(A, B[n]\big)$ is a finite R–module for every $n \in \mathbb{Z}$.
2. $\mathrm{Hom}\big(A, B[n]\big)$ vanishes for all but finitely many $n \in \mathbb{Z}$.

Definition

Let $\mathcal{T}$ be a triangulated category and let R be a noetherian ring. A functor $H : \mathcal{T} \longrightarrow R\text{–}\mathrm{Mod}$ is *finite* if, for every object $B \in \mathcal{T}$, the R–module $\oplus_n H^n(B)$ is finite.

Main Theorems

Theorem (1)

Let R be a noetherian ring and assume X is proper over R. Then the Yoneda functor

$$\mathbf{D}^b_{\mathbf{coh}}(X) \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\Big(\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big)$$

taking B to $\mathrm{Hom}(-,B)$ is fully faithful, and the essential image is the set of finite R–linear homological functors $H:\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}} \longrightarrow R\text{–}\mathrm{Mod}$.

Theorem (2)

Let R be a noetherian ring, assume X is proper over R, and suppose further that every irreducible, reduced closed subscheme $Z \subset X$ has a regular alteration. Then the Yoneda functor

$$\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\Big(\mathbf{D}^b_{\mathbf{coh}}(X)\ ,\ R\text{–}\mathrm{Mod}\Big)$$

taking A to $\mathrm{Hom}(A,-)$ is fully faithful, and the essential image is the set of finite R–linear homological functors $H : \mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}} \longrightarrow R\text{–}\mathrm{Mod}$.

Definition

Let $\mathcal{S}$ be a triangulated category. An object $G \in \mathcal{S}$ is a *classical generator* if any thick subcategory of $\mathcal{S}$ containing G is all of $\mathcal{S}$.

Counting the cost

We declare:

1. Shifting, forming finite direct sums, and forming direct summands is free.
2. Each time we form an extension, that is a triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$, then we must pay \$1 to obtain B from A and C.

Definition

Let $\mathcal{S}$ be a triangulated category. An object $G \in \mathcal{S}$ is a *strong generator* if there is a number $N > 0$ so that every object of $\mathcal{S}$ is obtainable from G at a cost of $\leq \$N$.

Theorem (3)

Let X be a quasicompact, separated scheme. The category $\mathbf{D}^{\mathrm{perf}}(X)$ has a strong generator if and only if X has a cover by open affine subsets $\mathrm{Spec}(R_i)$ with each R_i of finite global dimension.

Theorem (4)

Let X be a noetherian scheme, and suppose further that every irreducible, reduced closed subscheme $Z \subset X$ has a regular alteration. Then $\mathbf{D}^b_{\mathbf{coh}}(X)$ has a strong generator.

Theorem (5)

Let X be a quasicompact, separated scheme. Then the category $\mathbf{D_{qc}}(X)$ is approximable.

Where we're headed: formal definition of approximability

Let $\mathcal{T}$ be a triangulated category with coproducts. It is *approximable* if:

There exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and an integer $A > 0$ so that

- $G[A] \in \mathcal{T}^{\leq 0}$ and $\mathrm{Hom}\big(G[-A]\,,\,\mathcal{T}^{\leq 0}\big) = 0$.
- For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and $E \in \overline{\langle G \rangle}_A^{[-A,A]}$

Further background: compact generation, t–structures and the subcategories $\overline{\langle G\rangle}_A^{[-A,A]}$

Assume $\mathcal{T}$ is a triangulated category with coproducts.

An object $G \in \mathcal{T}$ is compact if $\mathrm{Hom}(G, -)$ commutes with coproducts.

The compact object $G \in \mathcal{T}$ generates $\mathcal{T}$ if the equivalent conditions below hold:

1. If $X \in \mathcal{T}$ is an object, and if $\mathrm{Hom}\big(G, X[n]\big) \cong 0$ for all $n \in \mathbb{Z}$, then $X \cong 0$.
2. If a triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ is closed in $\mathcal{T}$ under coproducts, and contains the object G, then $\mathcal{S} = \mathcal{T}$.

Example (the object $R \in \mathbf{D}(R)$)

The category $\mathbf{D}(R)$ has coproducts: a family of cochain complexes

$$\cdots \longrightarrow A_\lambda^{-2} \longrightarrow A_\lambda^{-1} \longrightarrow A_\lambda^{0} \longrightarrow A_\lambda^{1} \longrightarrow A_\lambda^{2} \longrightarrow \cdots$$

has coproduct

$$\cdots \longrightarrow \coprod_{\lambda\in\Lambda} A_\lambda^{-2} \longrightarrow \coprod_{\lambda\in\Lambda} A_\lambda^{-1} \longrightarrow \coprod_{\lambda\in\Lambda} A_\lambda^{0} \longrightarrow \coprod_{\lambda\in\Lambda} A_\lambda^{1} \longrightarrow \coprod_{\lambda\in\Lambda} A_\lambda^{2} \longrightarrow \cdots$$

If $R \in \mathbf{D}(R)$ stands for the cochain complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow R \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

then it can be shown that there is an isomorphism of functors $\mathrm{Hom}_{\mathbf{D}(R)}(R,-) \cong H^0(-)$. Hence the object $R \in \mathbf{D}(R)$ is compact.

Example (the object $R \in \mathbf{D}(R)$, continued)

If $X \in \mathbf{D}(R)$ is an object such that $H^n(X) \cong \mathrm{Hom}\big(R, X[n]\big) \cong 0$ for all $n \in \mathbb{Z}$, then X is acyclic. The cochain map $0 \longrightarrow X$ is an isomorphism in $\mathbf{D}(R)$. That is: $X \cong 0$.

Thus the object $R \in \mathbf{D}(R)$ is a compact generator.

Example (the standard t–structure on $\mathbf{D}(R)$)

We define two full subcategories of $\mathbf{D}(R)$:

- $$\mathbf{D}(R)^{\leq 0} = \{A \in \mathbf{D}(R) \mid H^i(A) = 0 \text{ for all } i > 0\}$$
- $$\mathbf{D}(R)^{\geq 0} = \{A \in \mathbf{D}(R) \mid H^i(A) = 0 \text{ for all } i < 0\}$$

Definition

A t–structure on a triangulated category $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ satisfying

- $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0} \subset \mathcal{T}^{\geq 0}[1]$
- $\mathrm{Hom}\Big(\mathcal{T}^{\leq 0}[1]\,,\ \mathcal{T}^{\geq 0}\Big) = 0$
- Every object $B \in \mathcal{T}$ admits a triangle $A \longrightarrow B \longrightarrow C \longrightarrow$ with $A \in \mathcal{T}^{\leq 0}[1]$ and $C \in \mathcal{T}^{\geq 0}$.

Notation

Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ and an integer $n \in \mathbb{Z}$ we define

$$\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n] \qquad \text{and} \qquad \mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$$

Definition (structure that's a formal consequence)

Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ on a triangulated category $\mathcal{T}$, we define

$$\mathcal{T}^{-} = \bigcup_{n \in \mathbb{N}} \mathcal{T}^{\leq n}, \qquad \mathcal{T}^{+} = \bigcup_{n \in \mathbb{N}} \mathcal{T}^{\geq -n},$$

$$\mathcal{T}^{\heartsuit} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}, \qquad \mathcal{T}^{b} = \mathcal{T}^{-} \cap \mathcal{T}^{+}$$

Projective resolutions

Suppose we are given an object $F^* \in \mathbf{D}(R)$, meaning a cochain complex

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^{0} \longrightarrow F^{1} \longrightarrow F^{2} \longrightarrow \cdots$$

Assume $F^* \in \mathbf{D}(R)^{\leq 0}$, meaning

$$H^i(F^*) = 0 \qquad \text{for all } i > 0.$$

Then F^* has a projective resolution. We can produce a cochain map

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^{0} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^{0} & \longrightarrow & F^{1} & \longrightarrow & F^{2} & \longrightarrow \cdots
\end{array}$$

inducing an isomorphism in cohomology, and so that the P^i are projective.

Projective resolutions—a different perspective

We have found in $\mathbf{D}(R)$ an isomorphism $P^* \longrightarrow F^*$. Now consider

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

This gives in $\mathbf{D}(R)$ triangles

$$E_n^* \longrightarrow F^* \longrightarrow D_n^* \longrightarrow$$

with $D_n^* \in \mathbf{D}(R)^{\leq -n-1}$ and E_n^* not too complicated.

Reminder of standard notation

Let $\mathcal{T}$ be a triangulated category, possibly with coproducts, and let $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ be full subcategories. We define the full subcategories

- $$\mathcal{A} * \mathcal{B} \quad = \quad \left\{ x \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{there exists a triangle } a \longrightarrow x \longrightarrow b \\ \text{with } a \in \mathcal{A},\ b \in \mathcal{B} \end{array} \right\}$$
- add($\mathcal{A}$): all finite coproducts of objects of $\mathcal{A}$. [slightly nonstandard]
- Assume $\mathcal{T}$ has coproducts. Define Add($\mathcal{A}$): all coproducts of objects of $\mathcal{A}$. [slightly nonstandard]
- smd($\mathcal{A}$): all direct summands of objects of $\mathcal{A}$.

Measuring the effort

Let $\mathcal{T}$ be a triangulated category, possibly with coproducts, let $\mathcal{A} \subset \mathcal{T}$ be a full subcategory and let $m \leq n$ be integers. We define the full subcategories

- $\mathcal{A}[m,n] = \cup_{i=m}^{n} \mathcal{A}[-i]$
- $\langle \mathcal{A} \rangle_1^{[m,n]} = \text{smd}\Big[\text{add}\big(\mathcal{A}[m,n]\big)\Big]$
- $\overline{\langle \mathcal{A} \rangle}_1^{[m,n]} = \text{smd}\Big[\text{Add}\big(\mathcal{A}[m,n]\big)\Big]$ [assumes coproducts exist]

Now let $\ell > 0$ be an integer, and assume $\langle \mathcal{A} \rangle_k^{[m,n]}$ and $\overline{\langle \mathcal{A} \rangle}_k^{[m,n]}$ have been defined for all $1 \leq k \leq \ell$. We continue with

- $\langle \mathcal{A} \rangle_{\ell+1}^{[m,n]} = \text{smd}\Big[\langle \mathcal{A} \rangle_1^{[m,n]} * \langle \mathcal{A} \rangle_\ell^{[m,n]}\Big]$
- $\overline{\langle \mathcal{A} \rangle}_{\ell+1}^{[m,n]} = \text{smd}\Big[\overline{\langle \mathcal{A} \rangle}_1^{[m,n]} * \overline{\langle \mathcal{A} \rangle}_\ell^{[m,n]}\Big]$ [assumes coproducts exist]

Example (back to $\mathbf{D}(R)$—the version with finite coproducts)

Let $\mathcal{A} = \{R\}$ be the full subcategory of $\mathbf{D}(R)$ with a single object. Then

- $\langle R \rangle_1^{[-n,0]}$: all isomorphs of complexes

$$\cdots \longrightarrow 0 \longrightarrow P^{-n} \xrightarrow{0} \cdots \xrightarrow{0} P^{-1} \xrightarrow{0} P^0 \longrightarrow 0 \longrightarrow \cdots$$

  with P^i finitely generated and projective.
- $\langle R \rangle_{n+1}^{[-n,0]}$: all isomorphs of complexes

$$\cdots \longrightarrow 0 \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0 \longrightarrow \cdots$$

  with P^i finitely generated and projective.

Example (still $\mathbf{D}(R)$—but now the infinite version)

Let $\mathcal{A} = \{R\}$ be the full subcategory of $\mathbf{D}(R)$ with a single object. Then

- $\overline{\langle R\rangle}_1^{[-n,0]}$: all isomorphs of complexes

$$\cdots \longrightarrow 0 \longrightarrow P^{-n} \xrightarrow{\ 0\ } \cdots \xrightarrow{\ 0\ } P^{-1} \xrightarrow{\ 0\ } P^0 \longrightarrow 0 \longrightarrow \cdots$$

  with P^i projective.
- $\overline{\langle R\rangle}_{n+1}^{[-n,0]}$: all isomorphs of complexes

$$\cdots \longrightarrow 0 \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0 \longrightarrow \cdots$$

  with P^i projective.

Definition (formal definition of approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is *approximable* if there exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and an integer $A > 0$ so that

- $G[A] \in \mathcal{T}^{\leq 0}$ and $\mathrm{Hom}\big(G[-A]\,,\,\mathcal{T}^{\leq 0}\big) = 0$.
- For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and $E \in \overline{\langle G \rangle}_A^{[-A,A]}$

Example (the category $\mathbf{D}(R)$)

Let R be a ring. The object $R \in \mathbf{D}(R)$ is a compact generator, the t–structure we take is the standard one, and we set $A = 1$. It's clear that $R[1] \in \mathbf{D}(R)^{\leq 0}$ and that $\mathrm{Hom}\big(R[-1], \mathbf{D}(R)^{\leq 0}\big) = 0$. Finally, given an object $F \in \mathbf{D}(R)^{\leq 0}$ we first replace F by a projective resolution, then form the triangle $E \longrightarrow F \longrightarrow D$ below

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P^{-n-1} & \longrightarrow & P^{-n} & \longrightarrow & \cdots & \longrightarrow & P^{-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

with $D \in \mathbf{D}(R)^{\leq -1}$ and $E \in \overline{\langle R \rangle}_1^{[0,0]} \subset \overline{\langle R \rangle}_1^{[-1,1]}$.

The main theorems—sources of more examples

1. If $\mathcal{T}$ has a compact generator G so that $\mathrm{Hom}\big(G, G[i]\big) = 0$ for all $i \geq 1$, then $\mathcal{T}$ is approximable.
2. Let X be a quasicompact, separated scheme. Then the category $\mathbf{D_{qc}}(X)$ is approximable.
3. [Joint with Jesse Burke and Bregje Pauwels]: Suppose we are given a recollement of triangulated categories

$$\mathcal{R} \leftleftarrows\!\!\!\rightarrow \mathcal{S} \leftleftarrows\!\!\!\rightarrow \mathcal{T}$$

with $\mathcal{R}$ and $\mathcal{T}$ approximable. Assume further that the category $\mathcal{S}$ is compactly generated, and any compact object $H \in \mathcal{S}$ has the property that $\mathrm{Hom}\big(H, H[i]\big) = 0$ for $i \gg 0$. Then the category $\mathcal{S}$ is also approximable.

The main theorems—applications

1. Let X be a quasicompact, separated scheme. The category $\mathbf{D}^{\mathrm{perf}}(X)$ is strongly generated if and only if X has an open cover by affine schemes $\mathrm{Spec}(R_i)$, with each R_i of finite global dimension.
2. Let X be a separated scheme, of finite type over an excellent scheme S of dimension ≤ 2. Then the category $\mathbf{D}^b_{\mathbf{coh}}(X)$ is strongly generated.
3. Let X be a scheme proper over a noetherian ring R. Then the Yoneda map

$$\mathbf{D}^b_{\mathbf{coh}}(X) \xrightarrow{\qquad \mathcal{Y} \qquad} \mathrm{Hom}_R\Big(\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big)$$

is fully faithful, and the essential image is the set of finite R–linear cohomological functors $H : \mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}} \longrightarrow R$–mod. An R–linear cohomological functor is *finite* if, for all objects $C \in \mathbf{D}^{\mathrm{perf}}(X)$, the R–module $\oplus_i H^i(C)$ is finite.

We should remind the reader what the terms used in the theorems mean.

Some old definitions

Let $\mathcal{S}$ be a triangulated category, and let $G \in \mathcal{S}$ be an object.

- G is a *classical generator* if $\mathcal{S} = \cup_n \langle G \rangle_n^{[-n,n]}$.
- G is a *strong generator* if there exists an integer $\ell > 0$ with $\mathcal{S} = \cup_n \langle G \rangle_\ell^{[-n,n]}$. The category $\mathcal{S}$ is *strongly generated* if there exists a strong generator $G \in \mathcal{S}$.

What was known

Theorem (application (3), reminder)

Let X be a scheme proper over a noetherian ring R. Then the Yoneda map

$$\mathbf{D}^b_{\mathbf{coh}}(X) \xrightarrow{\qquad \mathcal{Y} \qquad} \mathrm{Hom}_R\Big(\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big)$$

is fully faithful, and the essential image is the set of finite R–linear cohomological functors $H : \mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}} \longrightarrow R$–mod.

This means: the functor taking an object $B \in \mathbf{D}^b_{\mathbf{coh}}(X)$ to the functor $\mathcal{Y}(B) = \mathrm{Hom}(-, B)$, which we view as an R–linear functor $\mathbf{D}^{\mathrm{perf}}(X) \longrightarrow R\text{–}\mathrm{Mod}$, is fully faithful and has the image described.

What was known before

- If X is proper over R, if $A \in \mathbf{D}^{\mathrm{perf}}(X)$ and if $B \in \mathbf{D}^b_{\mathbf{coh}}(X)$, then

$$\mathrm{Hom}(A[i], B) \quad \cong \quad H^{-i}(A^\vee \otimes B)$$

  is a finite R–module for every i and vanishes outside a bounded range. This much was proved by Grothendieck in EGA.

  Rephrasing in terms of $\mathcal{Y}$: if $B \in \mathbf{D}^b_{\mathbf{coh}}(X)$ then $\mathcal{Y}(B) = \mathrm{Hom}(-, B)$ is a finite cohomological functor on $\mathbf{D}^{\mathrm{perf}}(X)$. This much has been known since the 1950s.

- **As long as R is a field** Bondal and Van den Bergh proved, in 2003, that every finite cohomological functor on $\mathbf{D}^{\mathrm{perf}}(X)$ is of the form $\mathcal{Y}(B) = \mathrm{Hom}(-, B)$ for some $B \in \mathbf{D}^b_{\mathbf{coh}}(X)$.

Theorem (application (1), reminder)

Let X be a quasicompact, separated scheme. The category $\mathbf{D}^{\mathrm{perf}}(X)$ is strongly generated if and only if X has an open cover by affine schemes $\mathrm{Spec}(R_i)$, with each R_i of finite global dimension.

Remark: if X is noetherian and separated, this simplifies to saying that $\mathbf{D}^{\mathrm{perf}}(X)$ is strongly generated if and only if X is regular and finite dimensional.

What was known before

The following special cases were known:

- If X is an affine scheme, the theorem goes back to a 1965 article by Max Kelly.
- If X is smooth over a field k, the theorem may be found in a 2003 article by Bondal and Van den Bergh.
- If X is regular and of finite type over a field, the theorem may be deduced from either a 2008 result of Rouquier, or a 2016 theorem of Orlov.

Theorem (application (2), reminder)

Let X be a separated scheme, of finite type over an excellent scheme S of dimension ≤ 2. Then the category $\mathbf{D}^b_{\mathbf{coh}}(X)$ is strongly generated.

What was known before

The following special cases were known:

- If X is regular and finite-dimensional the result follows easily from Application (1).
- If X is of finite type over a perfect field k, the theorem may be found in a 2008 article by Rouquier.
- The generalization to X of finite type over an arbitrary field may be found in a 2008 preprint by Keller and Van den Bergh. A different proof may be found in a 2010 paper by Lunts.
- Suppose X is affine—the question was studied in several papers by Takahashi and coathors. The union of the results says: $\mathbf{D}^b_{\mathbf{coh}}(X)$ is strongly generated as long as either X is of finite type over a field, or else it is the spectrum of an equicharacteristic complete local ring.

Proof of application (1)

The main point is that approximability allows us to easily reduce to Kelly's old theorem. We first remind the reader of Kelly's theorem and its proof.

Theorem (Kelly, 1965)

Suppose R is a ring, and $\mathbf{D}(R)$ its derived category. Let $n \geq 0$ be an integer, and let $F \in \mathbf{D}(R)$ be an object so that the projective dimension of $H^i(F)$ is $\leq n$ for all $i \in \mathbb{Z}$. Then $F \in \overline{\langle G \rangle}_{n+1}^{(-\infty,\infty)}$.

Before proving the theorem we remind the reader: any morphism $P \longrightarrow H^i(E)$ in $\mathbf{D}(R)$, for any projective R–module P and any $E \in \mathbf{D}(R)$, lifts uniquely to a cochain map

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & E^{i-2} & \longrightarrow & E^{i-1} & \longrightarrow & E^{i} & \longrightarrow & E^{i+1} & \longrightarrow & E^{i+2} & \longrightarrow & \cdots
\end{array}$$

Proof of Kelly's theorem. We prove this by induction on n. Suppose first that $n = 0$; hence $H^i(F)$ is projective for every $i \in \mathbb{Z}$. The identity map $H^i(F) \longrightarrow H^i(F)$ lifts to a cochain map

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^i(F) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{i-2} & \longrightarrow & F^{i-1} & \longrightarrow & F^i & \longrightarrow & F^{i+1} & \longrightarrow & F^{i+2} & \longrightarrow & \cdots
\end{array}$$

and when we combine, for every $i \in \mathbb{Z}$, we obtain a cochain map

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & H^{-2}(F) & \xrightarrow{0} & H^{-1}(F) & \xrightarrow{0} & H^0(F) & \xrightarrow{0} & H^1(F) & \xrightarrow{0} & H^2(F) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & F^2 & \longrightarrow & \cdots
\end{array}$$

This is an isomorphism in cohomology, hence an isomorphism in $\mathbf{D}(R)$.

Now suppose $n \geq 0$, and we know the result for every ℓ with $0 \leq \ell \leq n$. We wish to show it for $n+1$. Suppose therefore that we are given an object $F \in \mathbf{D}(R)$ with $H^i(F)$ of projective dimension $\leq n+1$ for every i. Choose for every i a projective module P^i and a surjection $P^i \longrightarrow H^i(F)$. Now form the corresponding cochain map

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P^i & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{i-2} & \longrightarrow & F^{i-1} & \longrightarrow & F^i & \longrightarrow & F^{i+1} & \longrightarrow & F^{i+2} & \longrightarrow & \cdots
\end{array}$$

and combine over i to form

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P^{-2} & \xrightarrow{0} & P^{-1} & \xrightarrow{0} & P^0 & \xrightarrow{0} & P^1 & \xrightarrow{0} & P^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & F^2 & \longrightarrow & \cdots
\end{array}$$

giving a map $P \longrightarrow F$, which we complete to a triangle $P \longrightarrow F \longrightarrow Q$. Clearly $P \in \overline{\langle R \rangle}_1^{(-\infty,\infty)}$ and $H^i(Q)$ is of projective dimension $\leq n$.

Lemma

Let X be a quasicompact, separated scheme, let $G \in \mathbf{D_{qc}}(X)$ be a compact generator, and let $u : U \longrightarrow X$ be an open immersion with U quasicompact. Then the object $\mathbf{R}u_*\mathcal{O}_U \in \mathbf{D_{qc}}(X)$ belongs to $\overline{\langle G\rangle}_n^{[-n,n]}$ for some integer $n > 0$.

Proof.

It is relatively easy to show that there exists an integer $\ell > 0$ with $\mathrm{Hom}\big(\mathbf{R}u_*\mathcal{O}_U\ ,\ \mathbf{D_{qc}}(X)^{\leq -\ell}\big) = 0$. By the approximability of $\mathbf{D_{qc}}(X)$ we may choose an integer n and a triangle $E \longrightarrow \mathbf{R}u_*\mathcal{O}_U \longrightarrow D$ with $D \in \mathbf{D_{qc}}(X)^{\leq -\ell}$ and $E \in \overline{\langle G\rangle}_n^{[-n,n]}$.

But the map $\mathbf{R}u_*\mathcal{O}_U \longrightarrow D$ must vanish by the choice of ℓ, making $\mathbf{R}u_*\mathcal{O}_U$ a direct summand of the object $E \in \overline{\langle G\rangle}_n^{[-n,n]}$. □

Sketch of how application (1) follows from the Lemma

Let X be a quasicompact, separated scheme. By hypothesis we may cover X by open subsets $U_i = \mathrm{Spec}(R_i)$ with each R_i of finite global dimension. By the quasicompactness we may choose finitely many U_i which cover. The Lemma tells us that we may choose a compact generator $G \in \mathbf{D_{qc}}(X)$ and an integer n so that $\mathbf{R}u_{i*}\mathcal{O}_{U_i} \in \overline{\langle G \rangle}_n^{[-n,n]}$ for every i in the finite set.

Put $\mathcal{G} = \{G[k],\ k \in \mathbb{Z}\}$. Then it certainly follows that $\mathbf{R}u_{i*}\mathcal{O}_{U_i} \in \overline{\langle \mathcal{G} \rangle}_n^{[0,0]}$.

Now put $\mathcal{U}_i = \{\mathcal{O}_{U_i}[n],\ n \in \mathbb{Z}\}$. Since R_i is of finite global dimension, Kelly's 1965 theorem tells us that we may choose an integer $\ell > 0$ so that $\mathbf{D_{qc}}(U_i) \in \overline{\langle \mathcal{U}_i \rangle}_\ell^{[0,0]}$. It follows that

$$\mathbf{R}u_{i*}\mathbf{D_{qc}}(U_i) \quad \in \quad \overline{\langle \mathbf{R}u_{i*}\mathcal{U}_i \rangle}_\ell^{[0,0]} \quad \subset \quad \overline{\langle \mathcal{G} \rangle}_{\ell n}^{[0,0]}$$

Sketch of how application (1) follows from the Lemma—continued

It's an exercise to show that $\mathbf{D_{qc}}(X)$ can be generated from the subcategories $\mathbf{R}u_{i*}\mathbf{D_{qc}}(U_i)$ in finitely many steps. Hence there exists an integer N with $\mathbf{D_{qc}}(X) = \overline{\langle \mathcal{G} \rangle}_N^{[0,0]}$.

We have proved a statement about $\mathbf{D_{qc}}(X)$, and $\mathbf{D}^{\mathrm{perf}}(X) \subset \mathbf{D_{qc}}(X)$ is the subcategory of compact objects. Standard compactness arguments give that $\mathbf{D}^{\mathrm{perf}}(X) = \langle \mathcal{G} \rangle_N^{[0,0]}$, which is Application (1).

Sketch of another consequence of the Lemma

Reminder of the technical terms

- Given a morphism of schemes $f : X \longrightarrow Y$, for any $x \in X$ there is an induced ring homomorphism $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ of the stalks. The map f is *of finite Tor-dimension at* x if $\mathcal{O}_{X,x}$ has a finite flat resolution over $\mathcal{O}_{Y,f(x)}$.
- The map f is *of finite Tor-dimension* if it is of finite Tor-dimension at every $x \in X$.
- The complex $C \in \mathbf{D_{qc}}(Y)$ is *of bounded-below Tor-amplitude* if, for every open immersion $u : U \longrightarrow Y$ with $U = \mathrm{Spec}(R)$ affine, the complex $u^*C \in \mathbf{D_{qc}}(U) \cong \mathbf{D}(R)$ is isomorphic to a bounded-below K–projective complex.

Theorem

Suppose $f : X \longrightarrow Y$ *is a separated morphism of quasicompact, quasiseparated schemes. Suppose* $\mathbf{R}f_* : \mathbf{D_{qc}}(X) \longrightarrow \mathbf{D_{qc}}(Y)$ *takes every perfect complex to a complex of bounded–below Tor-amplitude.*

Then f *is of finite Tor-dimension.*

What was known

If X and Y are noetherian and f is separated and of finite type, then the converse is due to Illusie, in a 1971 exposé in SGA6.

The first progress in the direction of the Theorem appeared in a 2007 article by Lipman and the author. That theorem only applied to proper maps f.

Sketch of proof

The question is obviously local in Y, hence we may assume Y is affine, hence separated. As f is separated we deduce that X must be separated.

It suffices to show that, for each open immersion $u : U \longrightarrow X$ with U affine, the composite $U \xrightarrow{u} X \xrightarrow{f} Y$ is of finite Tor-dimension. By the Lemma there exists a perfect complex $G \in \mathbf{D_{qc}}(X)$ and an integer $n > 0$ with $\mathbf{R}u_*\mathcal{O}_U \in \overline{\langle G \rangle}_n^{[-n,n]}$. Therefore

$$(fu)_*\mathcal{O}_U \quad \cong \quad \mathbf{R}(fu)_*\mathcal{O}_U \quad \cong \quad \mathbf{R}f_*\mathbf{R}u_*\mathcal{O}_U \quad \subset \quad \overline{\langle \mathbf{R}f_*G \rangle}_n^{[-n,n]}$$

But $\mathbf{R}f_*G$ is of bounded Tor-amplitude by hypothesis, and in forming $\overline{\langle \mathbf{R}f_*G \rangle}_n^{[-n,n]}$ we only allow $\mathbf{R}f_*G[i]$ with $-n \leq i \leq n$, coproducts, extensions and direct summands. Hence the objects of $\overline{\langle \mathbf{R}f_*G \rangle}_n^{[-n,n]}$ have Tor-amplitude uniformly bounded below.

Application (2) and its proof

Theorem (application (2), reminder)

Let X be a separated scheme, of finite type over an excellent scheme S of dimension ≤ 2. Then the category $\mathbf{D}^b_{\mathbf{coh}}(X)$ is strongly generated.

Sketch of proof

Application (1) gives the theorem in the special case where X is regular and finite dimensional. The idea is to reduce to this case.

Resolutions of singularities might look tempting, but in mixed characteristic they are known to exist only in low dimension. So instead we use de Jong's theorem, about the existence of regular alterations. Under the hypotheses every closed subscheme of X has a regular alteration. It turns out that the theorem can be deduced from this using induction on the dimension of X and two old theorems of Thomason's.

Now for the proof of Application (3). We first recall the statement:

Theorem (application (3)—reminder)

Let X be a scheme proper over a noetherian ring R. Then the Yoneda map

$$\mathbf{D}^b_{\mathbf{coh}}(X) \xrightarrow{\quad\mathcal{Y}\quad} \mathrm{Hom}_R\Big(\mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big)$$

is fully faithful, and the essential image is the set of finite R–linear cohomological functors $H : \mathbf{D}^{\mathrm{perf}}(X)^{\mathrm{op}} \longrightarrow R$–mod.

As it happens we prove a far more general result, and the discussion of this result brings us naturally to the structure that all approximable triangulated categories share. Let us begin in greater generality.

‘

Definition (equivalent t–structures)

Let $\mathcal{T}$ be a triangulated category, and let $\big(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0}\big)$ and $\big(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0}\big)$ be two t–structures on $\mathcal{T}$. We declare them *equivalent* if there exists an integer $A > 0$ with $\mathcal{T}_1^{\leq -A} \subset \mathcal{T}_2^{\leq 0} \subset \mathcal{T}_1^{\leq A}$.

‘

Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact generator. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ *generated by* G.

More precisely: $\mathcal{T}_G^{\leq 0} = \overline{\langle G \rangle}^{(-\infty,0]}$, and $\mathcal{T}^{\geq 0}$ is the orthogonal.

If G and H are two compact generators for $\mathcal{T}$, then the t–structures $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ and $\big(\mathcal{T}_H^{\leq 0}, \mathcal{T}_H^{\geq 0}\big)$ are equivalent.

We say that a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ is in the *preferred equivalence class* if it is equivalent to $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ for some compact generator G, hence for every compact generator.

‘

Given a t–structure $\left(\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0}\right)$ it is customary to define the categories

$$\mathcal{T}^- = \cup_n \mathcal{T}^{\leq n}\,, \qquad \mathcal{T}^+ = \cup_n \mathcal{T}^{\geq -n}\,, \qquad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$$

It's obvious that equivalent t–structures yield the same $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Now assume that $\mathcal{T}$ has coproducts and there exists a single compact generator G. Then there is a preferred equivalence class of t–structures, and a correponding preferred $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$. These are intrinsic, they're independent of any choice. In the remainder of the article we only consider the “preferred” $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ in the preferred equivalence class. The full subcategory $\mathcal{T}_c^-$ is defined by

$$\mathcal{T}_c^- \quad = \quad \left\{ F \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{For all integers } n > 0 \text{ there exists a triangle} \\ E \longrightarrow F \longrightarrow D \longrightarrow E[1] \\ \text{with } E \text{ compact and } D \in \mathcal{T}^{\leq -n-1} \end{array} \right\}$$

We furthermore define $\mathcal{T}_c^b = \mathcal{T}^b \cap \mathcal{T}_c^-$.

It's obvious that the category $\mathcal{T}_c^-$ is intrinsic. As $\mathcal{T}_c^-$ and $\mathcal{T}^b$ are both intrinsic, so is their intersection $\mathcal{T}_c^b$.

We have defined all this intrinsic structure, assuming only that $\mathcal{T}$ is a triangulated category with coproducts and with a single compact generator. In this generality we know that the subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are thick.

Proposition (a condition that guarantees the thickness of $\mathcal{T}_c^b, \mathcal{T}_c^-$)

If $\mathcal{T}$ has a compact generator G, such that $\mathrm{Hom}\big(G, G[n]\big) = 0$ for $n \gg 0$, then the subcategories $\mathcal{T}_c^-$ and $\mathcal{T}_c^b$ are thick.

If $\mathcal{T}$ is approximable

If $\mathcal{T}$ is approximable there are:

an integer $A > 0$, a compact generator $G \in \mathcal{T}$, and a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$

so that $\mathrm{Hom}\big(G[-A], \mathcal{T}^{\leq 0}\big) = 0$ and $G[A] \in \mathcal{T}^{\leq 0}$, hence $G[n] \in \mathcal{T}^{\leq 0}$ for all $n \geq A$.

Therefore $\mathrm{Hom}\big(G, G[n]\big) = 0$ for all $n \geq 2A$. The Proposition tells us that the categories $\mathcal{T}_c^-$ and $\mathcal{T}_c^b$ are thick.

It would be nice to be able to work out examples: what does all of this intrinsic structure come down to in special cases?

Proposition (the way to obtain some preferred t–structures)

Assume the category $\mathcal{T}$ is approximable. We recall part of the definition: the category $\mathcal{T}$ is approximable if it has a compact generator G, a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ and an integer $A > 0$ satisfying some properties.

Then any t–structure, which comes as part of a triad satisfying the properties, must be in the preferred equivalence class. Furthermore: for any compact generator G' and any t–structure $\big(\widetilde{\mathcal{T}}^{\leq 0}, \widetilde{\mathcal{T}}^{\geq 0}\big)$ in the preferred equivalence class, there is an integer $A' > 0$ so that the properties hold.

Corollary (the case of $\mathbf{D}(R)$)

We began with $\mathbf{D}(R)$ as our motivating example, and in particular proved $\mathbf{D}(R)$ approximable. The proof used the standard t–structure, which must therefore be in the preferred equivalence class.

the case of $\mathbf{D_{qc}}(X)$

Let X be a quasicompact, separated scheme. We have told the reader that that there is a proof showing $\mathcal{T} = \mathbf{D_{qc}}(X)$ approximable—the t–structure used in this proof happens to be the standard t–structure.

The Proposition informs us that the standard t–structure must belong to the preferred equivalence class.

Hence the categories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are the usual: we have $\mathcal{T}^- = \mathbf{D}^-_{\mathbf{qc}}(X)$, $\mathcal{T}^+ = \mathbf{D}^+_{\mathbf{qc}}(X)$ and $\mathcal{T}^b = \mathbf{D}^b_{\mathbf{qc}}(X)$.

What about $\mathcal{T}_c^-$ and $\mathcal{T}_c^b$?

Example ($\mathbf{D_{qc}}(X)$ when X is affine)

Let R be a ring. In the category $\mathcal{T} = \mathbf{D}(R)$, the subcategory $\mathcal{T}_c^-$ agrees with the $\mathbf{D}^-(R\text{–proj})$.

Now assume R is a commutative ring, and let $X = \mathrm{Spec}(R)$. Then the natural functor $\mathbf{D}(R) \longrightarrow \mathbf{D_{qc}}(X)$ is an equivalence of categories. Putting $\mathcal{T} = \mathbf{D_{qc}}(X) \cong \mathbf{D}(R)$, we learn that $\mathcal{T}_c^- = \mathbf{D}^-(R\text{–proj})$.

The trivial observation about $\mathbf{D_{qc}}(X)$

Now let X be any quasicompact, separated scheme. If $u : U \longrightarrow X$ is an open immersion, then the functor $u^* : \mathbf{D_{qc}}(X) \longrightarrow \mathbf{D_{qc}}(U)$ respects the standard t–structure and sends compact objects in $\mathbf{D_{qc}}(X)$ to compact objects in $\mathbf{D_{qc}}(U)$.

Hence $u^*\mathbf{D_{qc}}(X)^-_c \subset \mathbf{D_{qc}}(U)^-_c$. Thus every object in $\mathbf{D_{qc}}(X)^-_c$ must be "locally in $\mathbf{D}^-(R\text{–}\mathrm{proj})$"—for every open immersion $u : \mathrm{Spec}(R) \longrightarrow X$ we must have that $u^*\mathbf{D_{qc}}(X)^-_c \subset \mathbf{D}^-(R\text{–}\mathrm{proj})$.

The objects "locally in $\mathbf{D}^-(R\text{–}\mathrm{proj})$" were first studied by Illusie in SGA6, in 1971. They have a name, they are the pseudocoherent complexes.

Theorem ($\mathcal{T}_c^-$ for $\mathcal{T} = \mathbf{D_{qc}}(X)$)

Suppose X is a quasicompact, separated scheme. The objects of $\mathbf{D_{qc}}(X)_c^-$ are precisely the pseudocoherent complexes.

This theorem may be found in a 2007 article by Lipman and me.

Example (X noetherian, separated)

In this case pseudocoherence simplifies: we have $\mathbf{D_{qc}}(X)_c^- = \mathbf{D}^-_{\mathbf{coh}}(X)$.

The objects $F \in \mathbf{D}^-_{\mathbf{coh}}(X)$ are the complexes whose cohomology sheaves $\mathcal{H}^n(F)$ are coherent for all n, and vanish if $n \gg 0$.

And $\mathbf{D_{qc}}(X)_c^b$ is also explicit: it is the category $\mathbf{D}^b_{\mathbf{coh}}(X)$.

Theorem (the general version of application (3))

Let R be a noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}\big(G, G[n]\big)$ is a finite R–module for all $n \in \mathbb{Z}$.

1. *Consider the Yoneda functor*

$$\mathcal{Y} : \mathcal{T}_c^- \longrightarrow \mathrm{Hom}\big[(\mathcal{T}^c)^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\big].$$

   Then the functor $\mathcal{Y}$ is full, and the essential image of $\mathcal{Y}$ are the locally finite cohomological functors. A cohomological functor $H : \mathcal{T}^c \longrightarrow R\text{–}\mathrm{Mod}$ is locally finite if, for every object $A \in \mathcal{T}^c$, the R–module $H^i(A)$ is finite for every $i \in \mathbb{Z}$ and vanishes if $i \gg 0$.

2. *If we restrict the functor $\mathcal{Y}$ to the subcategory $\mathcal{T}_c^b \subset \mathcal{T}_c^-$, then $\mathcal{Y}$ is fully faithful and the essential image are the finite cohomological functors.*

References

Alexei I. Bondal and Michel Van den Bergh,
Generators and representability of functors in commutative and noncommutative geometry,
Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

Alexandre Grothendieck,
Éléments de géométrie algébrique III. Étude cohomologique des faisceaux cohérents I,
Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 425–511.

A. Johan de Jong,
Smoothness, semi-stability and alterations,
Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 51–93.

A. Johan de Jong,
Families of curves and alterations,
Ann. Inst. Fourier (Grenoble) **47** (1997), no. 2, 599–621.

Luc Illusie,
Conditions de finitude,
Théorie des intersections et théorème de Riemann-Roch, Springer-Verlag, Berlin, 1971, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6, Exposé III), pp. 222–273. Lecture Notes in Mathematics, Vol. 225.

G. Maxwell Kelly,
Chain maps inducing zero homology maps,
Proc. Cambridge Philos. Soc. **61** (1965), 847–854.

Joseph Lipman and Amnon Neeman,
Quasi-perfect scheme maps and boundedness of the twisted inverse image functor,
Illinois J. Math. **51** (2007), 209–236.

Valery A. Lunts,
Categorical resolution of singularities,
J. Algebra **323** (2010), no. 10, 2977–3003.

Amnon Neeman,
Strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$ *and* $\mathbf{D}^b_{\mathbf{coh}}(X)$,
preprint.

Amnon Neeman,
Triangulated categories with a single compact generator and a Brown representability theorem,
preprint.

Dmitri Orlov,
Smooth and proper noncommutative schemes and gluing of DG categories,
Adv. Math. **302** (2016), 59–105.

Raphaël Rouquier,
Dimensions of triangulated categories,
J. K-Theory **1** (2008), no. 2, 193–256.

Robert W. Thomason,
The classification of triangulated subcategories,
Compositio Math. **105** (1997), 1–27.

Robert W. Thomason and Thomas F. Trobaugh,
Higher algebraic K–theory of schemes and of derived categories,
The Grothendieck Festschrift (a collection of papers to honor Grothendieck's 60'th birthday), vol. 3, Birkhäuser, 1990, pp. 247–435.

Thank you!