Singular Hodge theory of matroids

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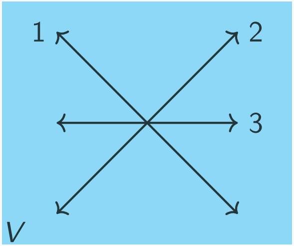
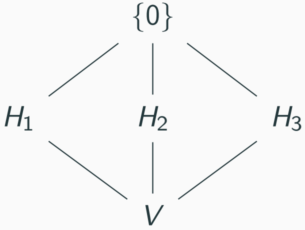
Hyperplane arrangements and flats





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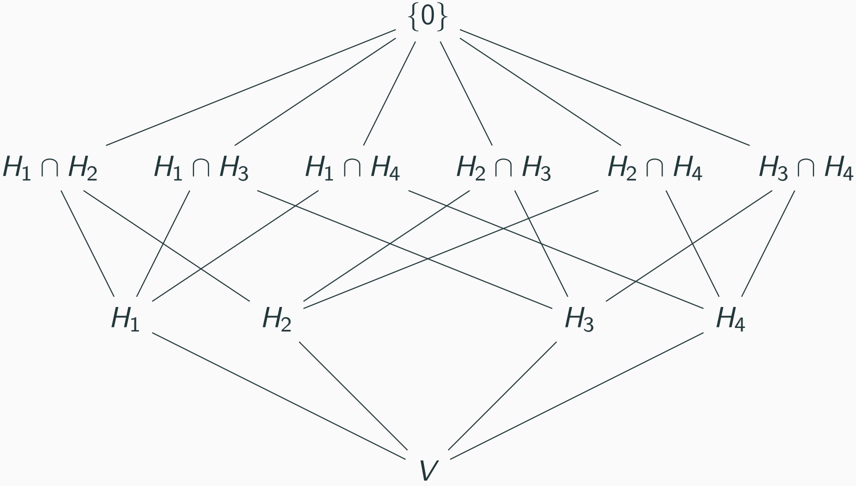

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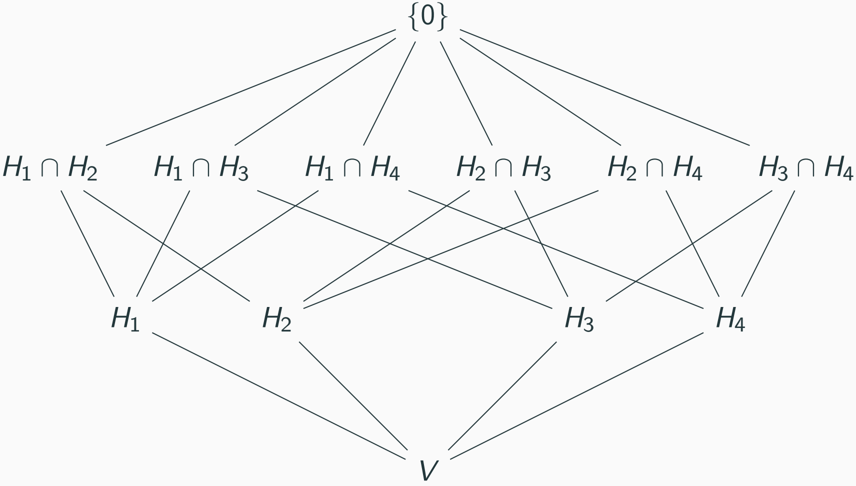

$6 \geq 4$

The Top Heavy Conjecture

Conjecture (Dowling–Wilson 1974)

For all $k \leq \frac{1}{2} \dim V$, we have

$$\#(\textit{flats of } \dim k) \geq \#(\textit{flats of } \operatorname{codim} k).$$

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Some geometry

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$$V \hookrightarrow \bigoplus_H V/H \cong \bigoplus_H \mathbb{A}^1 \subset \prod_H \mathbb{P}^1.$$

Let $Y := \overline{V} \subset \prod_H \mathbb{P}^1$.

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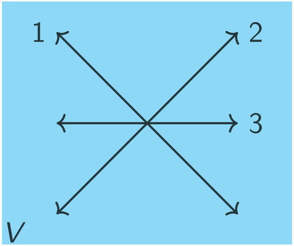

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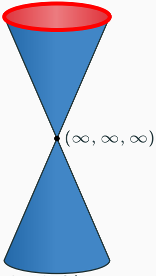

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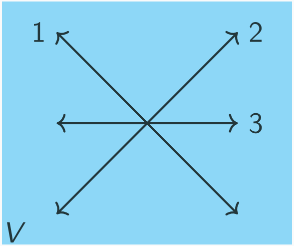

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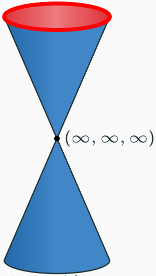

neighborhood of (∞, ∞, ∞)

Y has a stratification $Y = \coprod_F Y_F$ by affine spaces.

Affine pavings

The stratification by affine cells gives us two things:

1. dim $H^{2k}(Y) = \#$(flats of $\operatorname{codim} k$).
2. [Björner–Ekedahl 2009] There is an injection

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One property of intersection cohomology:

- $\mathrm{IH}^\bullet(Y)$ satisfies Hard Lefschetz (since Y is projective).

Proof of the Top Heavy Conjecture

Let $L \in H^2(Y)$ be an ample class. If $k \leq \frac{1}{2}\dim V$, then consider the following diagram.

$$\begin{tikzcd}
H^{2(\dim V-k)}(Y) \arrow[r, hook, "\text{B–E 09}"] & \mathrm{IH}^{2(\dim V-k)}(Y) \\
H^{2k}(Y) \arrow[u, "L^{2(\dim V-2k)}"] \arrow[r, hook, "\text{B–E 09}"'] & \mathrm{IH}^{2k}(Y) \arrow[u, "\cong"', "L^{2(\dim V-2k)}"]
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$\Longrightarrow$ Top Heavy Conjecture

Forward to matroids

The Top Heavy Conjecture makes sense for arbitrary matroids!

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Conjecture (Dowling–Wilson 1974)

Let M be an arbitrary matroid. For all $k \leq \frac{1}{2}\mathrm{rk}M$, we have

$$\#L(M)^{\mathrm{rk}M-k} \geq \#L(M)^k.$$

The semi-wonderful model (in progress by BHMPW)

Define a resolution

$$\widetilde{Y} \longrightarrow Y$$

by

1. first blowing up the point Y_V,
2. then the proper transforms of $\{Y_H\}$,
3. then the proper transforms of $\{Y_F\}$, where $\operatorname{codim} F = 2$, and so on...

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- [Huh–Wang 2017] There is a ring $B^\bullet(M)$ such that $B^\bullet(M) \cong H^\bullet(Y)$ when M is realizable.
- [Braden–Huh–M.–Proudfoot–Wang] There is a ring $A^\bullet(M)$ such that $A^\bullet(M) \cong H^\bullet(\widetilde{Y})$ when M is realizable.

Strategy for the proof (in progress by BHMPW)

Note that

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4. Run the same argument.

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$\Longrightarrow$ Top Heavy Conjecture for all matroids

The end

Thanks!