Gorenstein Projective Modules for the Working Algebraist

Xiuhua Luo

Nantong University, China

xiuhualuo@ntu.edu.cn

Maurice Auslander Distinguished Lectures and International Conference

April 26, 2018

Overview

1. Background
   - Definition and Properties
   - Applications
2. The explicit construction of Gorenstein projective modules
   - Upper Triangular Matrix Rings
   - Path Algebras of Acyclic Quivers
   - Tensor Products of algebras

Gorenstein projective modules (Enochs and Jenda 1995)

Let R be a ring. A module M is *Gorenstein projective*, if there exists a complete projective resolution

$$P^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

such that $M \cong \operatorname{Ker} d^0$.

Gorenstein projective modules (Enochs and Jenda 1995)

Let R be a ring. A module M is *Gorenstein projective*, if there exists a complete projective resolution

$$P^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

such that $M \cong \operatorname{Ker} d^0$.

Let $\mathcal{GP}(R)$ be the category of Gorenstein projective modules.

Background

- In 1967, M. Auslander introduced G-dimension zero modules over a Noetherian commutative local ring.

Background

- In 1967, M. Auslander introduced G-dimension zero modules over a Noetherian commutative local ring.
- In 1969, M. Auslander and M. Bridger generalized these modules to two-sided Noetherian ring.

Background

- In 1967, M. Auslander introduced G-dimension zero modules over a Noetherian commutative local ring.
- In 1969, M. Auslander and M. Bridger generalized these modules to two-sided Noetherian ring.
- Avramov, Buchweitz, Martsinkovsky and Reiten proved that a finitely generated module M over Noetherian ring R is Gorenstein projective if and only if G-$\dim_R M$=0.

Properties

Theorem (Henrik Holm 2004)

Let R be a non-trivial associative ring. Then $\mathcal{GP}(R)$ is projectively resolving. That is to say, $\mathcal{GP}(R)$ contains the projective modules and is closed under extensions, direct summands, kernels of surjections.

Properties

Theorem (Henrik Holm 2004)

Let R be a non-trivial associative ring. Then $\mathcal{GP}(R)$ is projectively resolving. That is to say, $\mathcal{GP}(R)$ contains the projective modules and is closed under extensions, direct summands, kernels of surjections.

Theorem

If R is a Gorenstein ring, then $\mathcal{GP}(R)$ is contravariantly finite [Enochs and Jenda 1995], thus it is functorially finite, and hence $\mathcal{GP}(R)$ has AR-seqs [Auslander and Smalϕ 1980].

Properties

Theorem (Henrik Holm 2004)

Let R be a non-trivial associative ring. Then $\mathcal{GP}(R)$ is projectively resolving. That is to say, $\mathcal{GP}(R)$ contains the projective modules and is closed under extensions, direct summands, kernels of surjections.

Theorem

If R is a Gorenstein ring, then $\mathcal{GP}(R)$ is contravariantly finite [Enochs and Jenda 1995], thus it is functorially finite, and hence $\mathcal{GP}(R)$ has AR-seqs [Auslander and Smalϕ 1980].

Theorem (Apostolos Beligiannis 2005)

Let R be an Artin Gorenstein ring, then $\mathcal{GP}(R)$ is a Frobenius category whose projective-injective objects are exactly all the projective R-modules.

Properties

Theorem (Henrik Holm 2004)

Let R be a non-trivial associative ring. Then $\mathcal{GP}(R)$ is projectively resolving. That is to say, $\mathcal{GP}(R)$ contains the projective modules and is closed under extensions, direct summands, kernels of surjections.

Theorem

If R is a Gorenstein ring, then $\mathcal{GP}(R)$ is contravariantly finite [Enochs and Jenda 1995], thus it is functorially finite, and hence $\mathcal{GP}(R)$ has AR-seqs [Auslander and Smalϕ 1980].

Theorem (Apostolos Beligiannis 2005)

Let R be an Artin Gorenstein ring, then $\mathcal{GP}(R)$ is a Frobenius category whose projective-injective objects are exactly all the projective R-modules.

Applications

- Singularity theory: $\underline{\mathcal{GP}}(R) \cong D_{sg}(R)$ as triangular categories,
  Buchweitz: when R is Gorenstein Noetherian ring;
  Happel: when R is Gorenstein algebra.
  Ringel and Pu Zhang: $\underline{\mathcal{GP}}(kQ \otimes_k k[x]/(X^2)) \cong D^b(kQ)/[1]$.

Applications

- Singularity theory: $\underline{\mathcal{GP}}(R) \cong D_{sg}(R)$ as triangular categories,
  Buchweitz: when R is Gorenstein Noetherian ring;
  Happel: when R is Gorenstein algebra.
  Ringel and Pu Zhang: $\underline{\mathcal{GP}}(kQ \otimes_k k[x]/(X^2)) \cong D^b(kQ)/[1]$.
- Tate cohomology theory: $\hat{Ext}^n_R(M, N) = \mathrm{H^n Hom_R(T, N)}$ where T is a complete projetive resolution in a complete resolution $T \xrightarrow{v} P \xrightarrow{\pi} M$ with v_n bijection when $n >> 0$. [Avramov and Martsinkovsky]

Applications

- Singularity theory: $\underline{\mathcal{GP}}(R) \cong D_{sg}(R)$ as triangular categories,
  Buchweitz: when R is Gorenstein Noetherian ring;
  Happel: when R is Gorenstein algebra.
  Ringel and Pu Zhang: $\underline{\mathcal{GP}}(kQ \otimes_k k[x]/(X^2)) \cong D^b(kQ)/[1]$.
- Tate cohomology theory: $\hat{Ext}^n_R(M, N) = \mathrm{H^n Hom_R(T, N)}$ where T is a complete projetive resolution in a complete resolution $T \xrightarrow{v} P \xrightarrow{\pi} M$ with v_n bijection when $n >> 0$. [Avramov and Martsinkovsky]
- the invariant subspaces of nilpotent operators:
  Ringel and Schmidmeier: $\{(V, U, T) \mid T : V \to V, T^6 = 0, U \subset V, T(U) \subset U\} = \mathcal{GP}(k[T]/(T^6) \otimes_k k(\bullet \to \bullet))$;
  Kussin, Lenzing and Meltzer showed a surpring link between singularity theory and the invariant subspace problem of nilpotent operators.
- $\cdots$

The explicit construction of Gorenstein projective modules

Let A and B be rings, M an $A-B-$bimodule, and $T := \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}$. Assume that T is an Artin algebra and consider finitely generated $T-$modules. A $T-$module can be identified with a triple $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi$, where $X \in A$-mod, $Y \in B$-mod, and $\phi : M \otimes_B Y \to X$ is an $A-$map. $\mathcal{G}p(T)$ is the category of finitely generated Gorenstein proj. $T-$modules.

The explicit construction of Gorenstein projective modules

Let A and B be rings, M an $A-B-$bimodule, and $T := \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}$. Assume that T is an Artin algebra and consider finitely generated $T-$modules. A $T-$module can be identified with a triple $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi$, where $X \in A$-mod, $Y \in B$-mod, and $\phi : M \otimes_B Y \to X$ is an $A-$map. $\mathcal{G}p(T)$ is the category of finitely generated Gorenstein proj. $T-$modules.

Theorem2.1 (P. Zhang 2013)

Let A and B be algebras and M a $A-B-$bimodule with $\mathrm{pdim_A M} < \infty$, $\mathrm{pdim M_B} < \infty$, $T := \begin{pmatrix} A & {}_AM_B \\ 0 & B \end{pmatrix}$. Then $\begin{pmatrix} X \\ Y \end{pmatrix}_\phi \in \mathcal{G}p(T)$ if and only if $\phi : M \otimes_B Y \to X$ is an injective A-map, $\mathrm{Coker}\,\phi \in \mathcal{G}p(A)$ and $Y \in \mathcal{G}p(B)$.

Let $Q = (Q_0, Q_1, s, e)$ be a finite acyclic quiver, k a field, A a f. d. k-algebra. Label the vertices as $1, 2, \cdots, n$ such that for each arrow $\alpha, s(\alpha) > e(\alpha)$. Then $A \otimes_k kQ$ is equivalent to an upper triangular algebra.

Let $Q = (Q_0, Q_1, s, e)$ be a finite acyclic quiver, k a field, A a f. d. k-algebra. Label the vertices as $1, 2, \cdots, n$ such that for each arrow $\alpha, s(\alpha) > e(\alpha)$. Then $A \otimes_k kQ$ is equivalent to an upper triangular algebra.

Theorem 2.2 (joint with P.Zhang 2013)

Let Q be a finite acyclic quiver, and A a finite dimensional algebra over a field k. Let $X = (X_i,\ X_\alpha)$ be a representation of Q over A. Then X is Gorenstein projective if and only if X is separated monic, and $\forall i \in Q_0$, $X_i \in \mathcal{G}p(A),\ X_i/(\sum_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \operatorname{Im} X_\alpha) \in \mathcal{G}p(A)$.

Let $Q = (Q_0, Q_1, s, e)$ be a finite acyclic quiver, k a field, A a f. d. k-algebra. Label the vertices as $1, 2, \cdots, n$ such that for each arrow $\alpha, s(\alpha) > e(\alpha)$. Then $A \otimes_k kQ$ is equivalent to an upper triangular algebra.

Theorem 2.2 (joint with P.Zhang 2013)

Let Q be a finite acyclic quiver, and A a finite dimensional algebra over a field k. Let $X = (X_i,\ X_\alpha)$ be a representation of Q over A. Then X is Gorenstein projective if and only if X is separated monic, and $\forall i \in Q_0$, $X_i \in \mathcal{G}p(A),\ X_i/(\sum\limits_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} \operatorname{Im} X_\alpha) \in \mathcal{G}p(A)$.

Defintion 2.3 separated monic representation

A representation $X = (X_i, X_\alpha)$ of Q over A is *separated monic,* if for each $i \in Q_0$, the A-map $\bigoplus\limits_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \overset{(X_\alpha)}{\to} X_i$ is injective.

In fact, let $\Lambda = A \otimes_k kQ, D = \mathrm{Hom_k}(-, \mathrm{k}),\ \mathrm{S_i}$ is a simple left kQ-module,

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} e_{s(\alpha)} kQ \overset{(\alpha.)}{\to} e_i kQ \to D(S_i) \to 0, \text{exact}$$

In fact, let $\Lambda = A \otimes_k kQ, D = \mathrm{Hom_k}(-, \mathrm{k}),\ \mathrm{S_i}$ is a simple left kQ-module,

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} e_{s(\alpha)} kQ \overset{(\alpha.)}{\to} e_i kQ \to D(S_i) \to 0, \text{exact}$$

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} A \otimes e_{s(\alpha)} kQ \overset{(1 \otimes \alpha.)}{\to} A \otimes e_i kQ \to A \otimes D(S_i) \to 0, \text{exact}$$

In fact, let $\Lambda = A \otimes_k kQ, D = \mathrm{Hom_k}(-, \mathrm{k}),\ \mathrm{S_i}$ is a simple left kQ-module,

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} e_{s(\alpha)} kQ \overset{(\alpha.)}{\to} e_i kQ \to D(S_i) \to 0, \text{exact}$$

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} A \otimes e_{s(\alpha)} kQ \overset{(1 \otimes \alpha.)}{\to} A \otimes e_i kQ \to A \otimes D(S_i) \to 0, \text{exact}$$

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} (1 \otimes e_{s(\alpha)})\Lambda \overset{(1 \otimes \alpha.)}{\to} (1 \otimes e_i)\Lambda \to A \otimes D(S_i) \to 0, \text{exact}$$

In fact, let $\Lambda = A \otimes_k kQ, D = \mathrm{Hom_k}(-, \mathrm{k}),\ \mathrm{S_i}$ is a simple left kQ-module,

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} e_{s(\alpha)} kQ \overset{(\alpha.)}{\to} e_i kQ \to D(S_i) \to 0, \text{exact}$$

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} A \otimes e_{s(\alpha)} kQ \overset{(1 \otimes \alpha.)}{\to} A \otimes e_i kQ \to A \otimes D(S_i) \to 0, \text{exact}$$

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} (1 \otimes e_{s(\alpha)})\Lambda \overset{(1 \otimes \alpha.)}{\to} (1 \otimes e_i)\Lambda \to A \otimes D(S_i) \to 0, \text{exact}$$

$$0 \to \bigoplus_{\substack{\alpha \in Q_1 \\ e(\alpha)=i}} X_{s(\alpha)} \overset{(X_\alpha)}{\to} X_i \to (A \otimes D(S_i)) \otimes_\Lambda X \to 0 \quad (*)$$

$(*)$ is exact if and only if $\bigoplus_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} X_{s(\alpha)} \overset{(X_\alpha)}{\to} X_i$ is injective if and only if $Tor_i^{\Lambda}(A\otimes_k D(S_i), X) = 0$ for all $i \geq 1$ and all simple left kQ-modules S_i.

$(*)$ is exact if and only if $\bigoplus_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} X_{s(\alpha)} \xrightarrow{(X_\alpha)} X_i$ is injective if and only if $Tor_i^{\Lambda}(A\otimes_k D(S_i), X) = 0$ for all $i \geq 1$ and all simple left kQ-modules S_i.

Definion 2.4 (Generalized) separated monic representation

Let k be a field, A and B finite dimensional k-algebras, $\Lambda := A\otimes_k B$. A left Λ-module X is called a *(generalized) separated monic representation* of B over A, if

$$Tor_i^{\Lambda}(A\otimes_k D(S), X) = 0$$

for all $i \geq 1$ and all simple left B-modules S.

$smon(B, A)$: the category of separated monic representation of B over A.

Define

$$smon(B, \mathcal{G}p(A)) := \{X \in smon(B, A) \mid (A \otimes_k V) \otimes_\Lambda X \in \mathcal{G}p(A),\ \forall\ V_B\}.$$

Define

$$smon(B, \mathcal{G}p(A)) := \{X \in smon(B, A) \mid (A \otimes_k V) \otimes_\Lambda X \in \mathcal{G}p(A),\ \forall\ V_B\}.$$

Propersition 2.5

Let A and B be f. d. k-algebras. Then $smon(B, \mathcal{G}p(A)) \subset \mathcal{G}p(\Lambda)$.

Define

$$smon(B, \mathcal{G}p(A)) := \{X \in smon(B, A) \mid (A \otimes_k V) \otimes_\Lambda X \in \mathcal{G}p(A),\ \forall\ V_B\}.$$

Propersition 2.5

Let A and B be f. d. k-algebras. Then $smon(B, \mathcal{G}p(A)) \subset \mathcal{G}p(\Lambda)$.

Question: When does $\mathcal{G}p(\Lambda)$ coincide with $smon(B, \mathcal{G}p(A))$?

Define

$$smon(B, \mathcal{G}p(A)) := \{X \in smon(B, A) \mid (A \otimes_k V) \otimes_\Lambda X \in \mathcal{G}p(A),\ \forall\ V_B\}.$$

Propersition 2.5

Let A and B be f. d. k-algebras. Then $smon(B, \mathcal{G}p(A)) \subset \mathcal{G}p(\Lambda)$.

Question: When does $\mathcal{G}p(\Lambda)$ coincide with $smon(B, \mathcal{G}p(A))$?

Theorem 2.6 (joint with W. Hu, B. Xiong and G. Zhou 2018)

- Suppose that B is Gorenstein. Then $smon(B, \mathcal{G}p(A)) = \mathcal{G}p(\Lambda)$ if and only if $\mathrm{gl.dim(B)} < \infty$.

Define

$$smon(B, \mathcal{G}p(A)) := \{X \in smon(B, A) \mid (A \otimes_k V) \otimes_\Lambda X \in \mathcal{G}p(A),\ \forall\ V_B\}.$$

Propersition 2.5

Let A and B be f. d. k-algebras. Then $smon(B, \mathcal{G}p(A)) \subset \mathcal{G}p(\Lambda)$.

Question: When does $\mathcal{G}p(\Lambda)$ coincide with $smon(B, \mathcal{G}p(A))$?

Theorem 2.6 (joint with W. Hu, B. Xiong and G. Zhou 2018)

- Suppose that B is Gorenstein. Then $smon(B, \mathcal{G}p(A)) = \mathcal{G}p(\Lambda)$ if and only if $\mathrm{gl.dim(B)} < \infty$.
- Suppose that A is Gorenstein. Then $smon(B, \mathcal{G}p(A)) = \mathcal{G}p(\Lambda)$ if and only if B is CM-free.

Via filtration categories

$$\mathcal{G}p(A) \otimes \mathcal{G}p(B) := \{X \otimes_k Y \in A \otimes_k B - \mathrm{mod} \mid \mathrm{X} \in \mathcal{G}\mathrm{p}(\mathrm{A}),\ \mathrm{Y} \in \mathcal{G}\mathrm{p}(\mathrm{B})\}$$

$$\widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B)) \subset \mathcal{G}p(A \otimes_k B)$$

Via filtration categories

$$\mathcal{G}p(A) \otimes \mathcal{G}p(B) := \{X \otimes_k Y \in A \otimes_k B - \mathrm{mod} \mid \mathrm{X} \in \mathcal{G}\mathrm{p}(\mathrm{A}),\ \mathrm{Y} \in \mathcal{G}\mathrm{p}(\mathrm{B})\}$$

$$\widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B)) \subset \mathcal{G}p(A \otimes_k B)$$

Quenstion: Does $\mathcal{G}p(A \otimes_k B)$ coincide with $\widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B))$?

Via filtration categories

$$\mathcal{G}p(A) \otimes \mathcal{G}p(B) := \{X \otimes_k Y \in A \otimes_k B - \mathrm{mod} \mid \mathrm{X} \in \mathcal{G}\mathrm{p}(\mathrm{A}),\ \mathrm{Y} \in \mathcal{G}\mathrm{p}(\mathrm{B})\}$$

$$\widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B)) \subset \mathcal{G}p(A \otimes_k B)$$

Quenstion: Does $\mathcal{G}p(A \otimes_k B)$ coincide with $\widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B))$?

Theorem 2.7 (joint with W. Hu, B. Xiong and G. Zhou 2018)

- Let A and B be Gorenstein algebras. Assume that k is a splitting field for A or B. Then $\mathcal{G}p(A \otimes_k B) = \widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B))$.

Via filtration categories

$$\mathcal{G}p(A) \otimes \mathcal{G}p(B) := \{X \otimes_k Y \in A \otimes_k B - \mathrm{mod} \mid \mathrm{X} \in \mathcal{G}\mathrm{p}(\mathrm{A}),\ \mathrm{Y} \in \mathcal{G}\mathrm{p}(\mathrm{B})\}$$

$$\widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B)) \subset \mathcal{G}p(A \otimes_k B)$$

Quenstion: Does $\mathcal{G}p(A \otimes_k B)$ coincide with $\widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B))$?

Theorem 2.7 (joint with W. Hu, B. Xiong and G. Zhou 2018)

- Let A and B be Gorenstein algebras. Assume that k is a splitting field for A or B. Then $\mathcal{G}p(A \otimes_k B) = \widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B))$.
- Let A be an algebra, and let B be a upper triangular algebra such that k is a splitting field for B. Then $\mathcal{G}p(A \otimes_k B) = \widetilde{filt}(\mathcal{G}p(A) \otimes \mathcal{G}p(B))$.

References

- M. Auslander, M. Bridger, Stable module theory, Mem. Amer. Math. Soc. 94., Amer. Math. Soc., Providence, R.I., 1969.
- L. L. Avramov, A. Martsinkovsky, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension, Proc. London Math. Soc. 85(3)(2002), 393-440.
- M. Auslander, I. Reiten, Applications of contravariantly finite subcategories, Adv. Math. 86(1991), 111-152.
- A. Beligiannis, Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras, J. Algebra 288(1)(2005), 137-211.
- R.-O. Buchweitz, Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, Unpublished manuscript, Hamburg (1987), 155pp.
- E. E. Enochs, O. M. G. Jenda, Gorenstein injective and projective modules, Math. Z. 220(4)(1995), 611-633.

References

- E. E. Enochs, O. M. G. Jenda, Relative homological algebra, De Gruyter Exp. Math. 30. Walter De Gruyter Co., 2000.
- D. Happel, On Gorenstein algebras, in: Representation theory of finite groups and finite-dimensional algebras, Prog. Math. 95, 389-404, Birkhaüser, Basel, 1991.
- H. Holm, Gorenstein homological dimensions, J. Pure Appl. Algebra 189(1-3) (2004), 167-193.
- W. Hu, X. Luo, B. Xiong, G. Zhou, Gorenstein projective bimodules via monomorphism categories and filtration categories, to appear in J. Pure Appl. Algebra.
- D. Kussin, H. Lenzing, H. Meltzer, Nilpotent operators and weighted projective lines, J. Reine Angew. Math. 685(2013), 33-71.
- Z. Li, P. Zhang, A construction of Gorenstein-projective modules, J. Algebra 323 (2010), 1802-1812.

References

- C. M. Ringel, M. Schmidmeier, Submodules categories of wild representation type, J. Pure Appl. Algebra 205(2)(2006), 412-422.
- C. M. Ringel, M. Schmidmeier, Invariant subspaces of nilpotent operators I, J. rein angew. Math. 614 (2008), 1-52.
- C. M. Ringel, P. Zhang, Representations of quivers over the algebras of dual numbers, J. Algebra 475 (2017), 327-360.
- D. Simson, Representation types of the category of subprojective representations of a finite poset over $K[t]/(t^m)$ and a solution of a Birkhoff type problem, J. Algebra 311(2007), 1-30.
- B. Xiong, P. Zhang, Gorenstein-projective modules over triangular matrix Artin algebras, Algebra Colloquium 23(2016), 97-104.
- P. Zhang, Gorenstein-projective modules and symmetric recollement, J. Algebra 388(2013), 65-80.

Thank You!