A combinatorial Fourier transform for quiver representation varieties in type A

Pramod N. Achar, Maitreyee Kulkarni, and Jacob P. Matherne
Louisiana State University and University of Massachusetts Amherst

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Goal

Consider the quiver $\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$.

Notation:

1. $E(\mathbf{w})$ - space of representations for dimension vector $\mathbf{w} = (w_1, \ldots, w_n)$
2. $G(\mathbf{w}) = \mathbf{GL}(w_1) \times \cdots \times \mathbf{GL}(w_n)$
3. $\mathbf{w}^* = (w_n, \ldots, w_1)$ - the reverse dimension vector

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Can we give a combinatorial description of the Fourier–Sato transform:

$$\begin{array}{rcl} D^{\mathrm{b}}_{G(\mathbf{w})}(E(\mathbf{w})) & \xrightarrow{\mathbb{T}} & D^{\mathrm{b}}_{G(\mathbf{w}^*)}(E(\mathbf{w}^*)) \\ \mathcal{F} & \longmapsto & q_{2!}q_1^*(\mathcal{F})[\dim E(\mathbf{w})] \end{array}$$

for simple perverse sheaves $\mathcal{F}$?

Outline

Quiver representation varieties

Some combinatorics

Fourier–Sato transform

Combinatorial Fourier transform

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Quiver representations

Consider the type A_n equioriented quiver

$$Q_n = \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet.$$

A **quiver representation** is:

- A finite-dimensional $\mathbb{C}$-vector space M_i for each vertex.
- A linear map x_i for each arrow.

$$M_1 \xrightarrow{x_1} M_2 \xrightarrow{x_2} \cdots \xrightarrow{x_{n-1}} M_n$$

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$\mathrm{Rep}(Q_n)$ - abelian category of finite-dimensional complex representations of Q_n

Quiver representation varieties

Fix a dimension vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)$.

A **quiver representation variety** $E(\mathbf{w})$ is the space of all quiver representations for a fixed dimension vector $\mathbf{w}$.

Note that $E(\mathbf{w})$ is an affine variety:

$$E(\mathbf{w}) \simeq \mathbb{A}^{w_1 w_2 + w_2 w_3 + \cdots + w_{n-1} w_n}.$$

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$G(\mathbf{w}) = \mathbf{GL}(w_1) \times \cdots \times \mathbf{GL}(w_n)$ acts on $E(\mathbf{w})$ by

$$(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_{n-1}) = (g_2 x_1 g_1^{-1}, \ldots, g_n x_{n-1} g_{n-1}^{-1})$$

giving it a stratification by orbits.

Note that two points $x, y \in E(\mathbf{w})$ are in the same $G(\mathbf{w})$-orbit if and only if they are isomorphic objects of $\mathrm{Rep}(Q_n)$.

Classifying the orbits

Theorem (Gabriel's Theorem)

There is a bijection

$\{\textit{indec. objects in } \mathrm{Rep}(Q_n)\}/\sim \overset{1-1}{\longleftrightarrow} \{\textit{pos. roots for } A_n \textit{ root system}\}.$

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To an indecomposable representation

$$R_{ij} = 0 \to \cdots \to 0 \to \underset{\text{vertex } i}{\mathbb{C}} \xrightarrow{\text{id}} \cdots \xrightarrow{\text{id}} \underset{\text{vertex } j}{\mathbb{C}} \to 0 \to \cdots \to 0.$$

we associate its dimension vector, the positive root

$$\gamma_{ij} = (0, \ldots, 0, \underset{\text{position } i}{1}, \ldots, \underset{\text{position } j}{1}, 0, \ldots, 0).$$

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Corollary

There is a bijection

$$\{G(\mathbf{w})\textit{-orbits in } E(\mathbf{w})\} \xleftrightarrow{1-1} B(\mathbf{w}) := \{b_{ij} \mid \sum b_{ij}\gamma_{ij} = \mathbf{w}\}.$$

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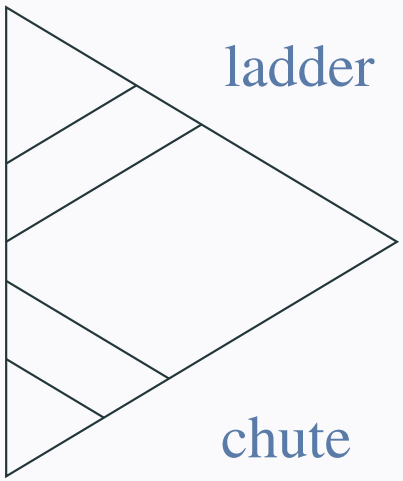
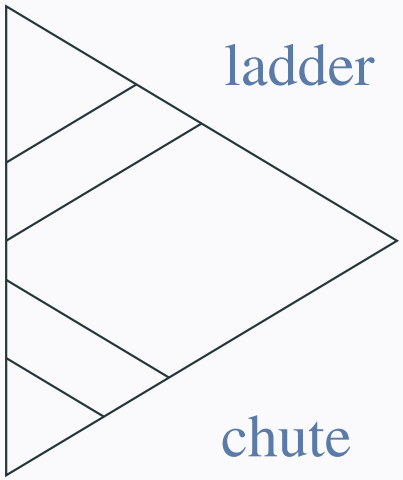
Triangular arrays



Define the set $\mathbf{P}(\mathbf{w})$ of triangular arrays of nonnegative integers such that:

- $\forall j$, the entries in the jth chute sum to w_j.
- Ladders are weakly decreasing.

Triangular arrays



Define the set $\mathbf{P}(\mathbf{w})$ of triangular arrays of nonnegative integers such that:

- $\forall j$, the entries in the jth chute sum to w_j.
- Ladders are weakly decreasing.

For $\mathbf{w} = (1, 1, 2)$, $\begin{array}{ccc} 1 & & \\ & 0 & \\ 0 & & 0 \\ & 1 & \\ 2 & & \end{array} \in \mathbf{P}(\mathbf{w})$

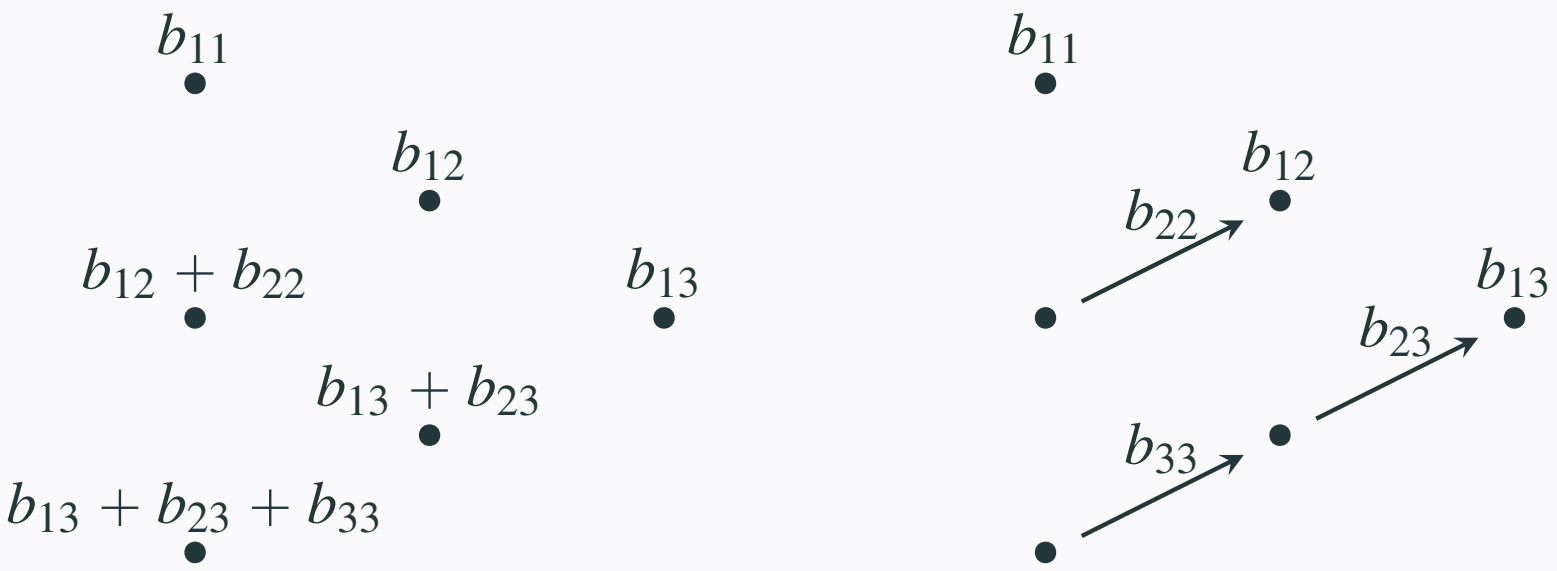
We will write y_{ij} for the entry in the ith chute and j^{th} column.

Classifying the orbits combinatorially

Lemma (Achar–K.–Matherne)

There is a bijection

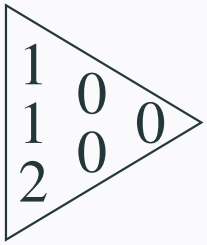
$$B(\mathbf{w}) := \{b_{ij} \mid \sum b_{ij}\gamma_{ij} = \mathbf{w}\} \xleftrightarrow{1-1} \mathbf{P}(\mathbf{w}).$$



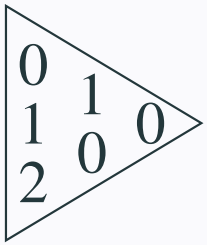
Running Example (A_3)

Let $\mathbf{w} = (1, 1, 2)$.

$$\mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C}^2$$



$$\mathbb{C} \xrightarrow{\text{rank } 1} \mathbb{C} \xrightarrow{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \mathbb{C}^2$$



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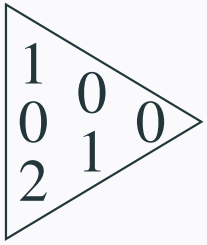

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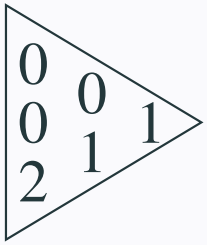

Partial orders on $\mathbf{P}(\mathbf{w})$

If $Y \in \mathbf{P}(\mathbf{w})$, we write $\mathcal{O}_Y$ for the corresponding $G(\mathbf{w})$-orbit in $E(\mathbf{w})$.

Let $Y, Y' \in \mathbf{P}(\mathbf{w})$.

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Geometric partial order on $\mathbf{P}(\mathbf{w})$:

$$Y \leqslant_{\mathbf{g}} Y' \qquad \text{if and only if} \qquad \mathcal{O}_Y \subset \overline{\mathcal{O}_{Y'}}.$$

Combinatorial partial order on $\mathbf{P}(\mathbf{w})$:

$$Y \leqslant_{\mathbf{c}} Y' \qquad \text{if for all } i \text{ and } j, \qquad \sum_{k=1}^{j} y_{ik} \geqslant \sum_{k=1}^{j} y'_{ik}.$$

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Theorem (Achar–K.–Matherne)

The geometric and combinatorial partial orders coincide.

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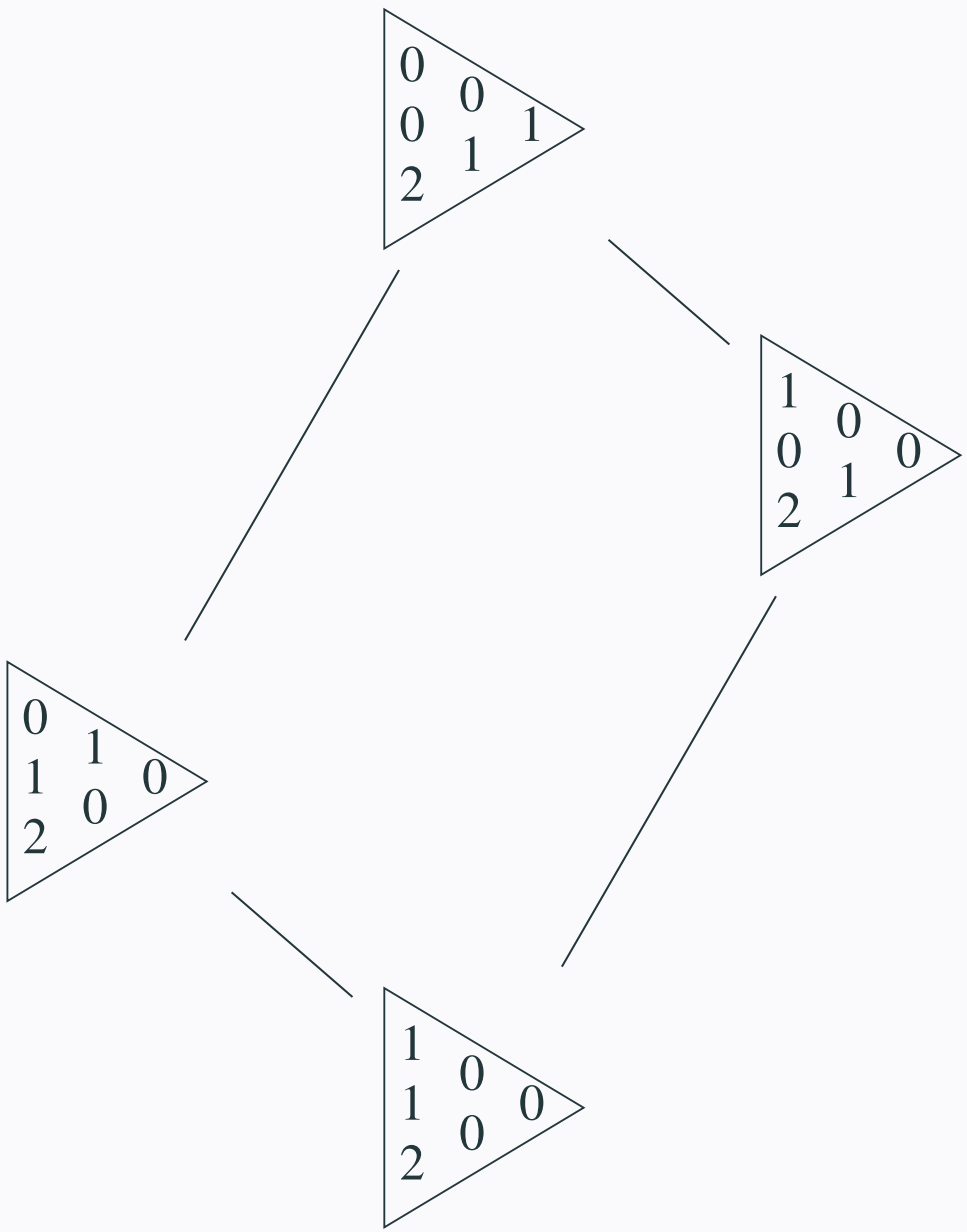

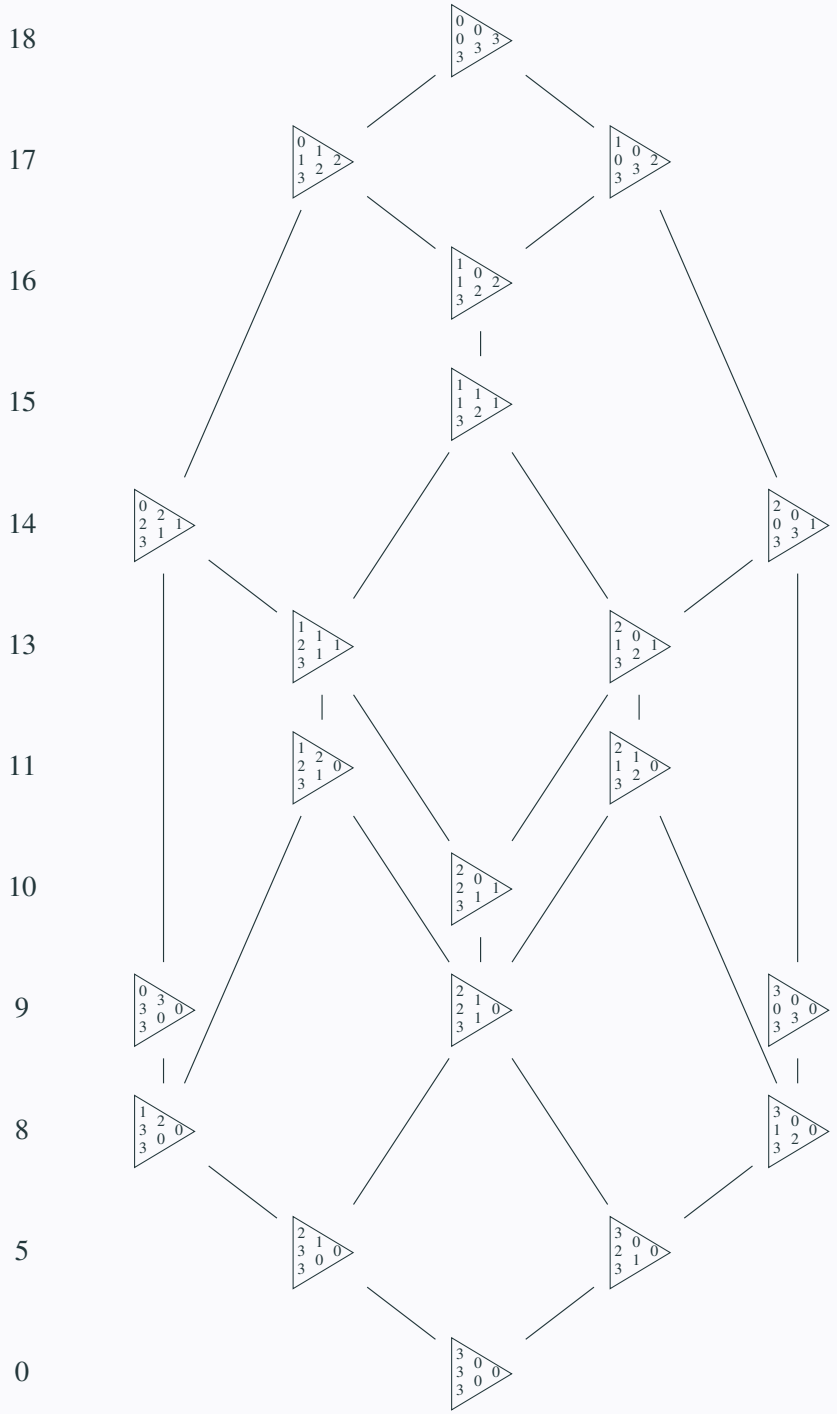
18
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Some observations from the combinatorics

Let $Y \in \mathbf{P}(\mathbf{w})$. Denote by $M(Y)$ a representation in the orbit $\mathcal{O}_Y$.

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Theorem (Achar–K.–Matherne)

1. $$\dim \mathcal{O}_Y = \sum_{\substack{1 \leqslant i \leqslant n-1 \\ 1 \leqslant j < k \leqslant n-i+1}} y_{ij} y_{ik} + \sum_{\substack{1 \leqslant i \leqslant n-1 \\ 1 \leqslant j < k \leqslant n-i+1}} y_{i+1,j} y_{ik}.$$

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2. *$M(Y)$ is an injective object in $\mathrm{Rep}(Q_n)$ if and only if Y is constant along ladders.*
3. *$M(Y)$ is a projective object in $\mathrm{Rep}(Q_n)$ if and only if Y has nonzero entries only in the last ladder.*

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Some basics about perverse sheaves

Perverse sheaves - complexes of sheaves that encode information about the singularities of a space (intersection cohomology)

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So, get a bijection:

$$\begin{array}{rcl} \mathbf{P}(\mathbf{w}) & \overset{1-1}{\longleftrightarrow} & \{\text{simple perverse sheaves on } E(\mathbf{w})\}. \\ Y & \longmapsto & \mathrm{IC}(\mathcal{O}_Y) \end{array}$$

Fourier–Sato transform

Can we give a combinatorial description of the Fourier–Sato transform:

$$\begin{array}{rcl} D^{\mathrm{b}}_{G(\mathbf{w})}(E(\mathbf{w})) & \xrightarrow{\mathbb{T}} & D^{\mathrm{b}}_{G(\mathbf{w}^*)}(E(\mathbf{w}^*)) \\ \mathcal{F} & \longmapsto & \mathcal{F}^\wedge[\dim E(\mathbf{w})] \end{array}$$

for simple perverse sheaves $\mathcal{F}$?

Running example

$\mathbf{P}(\mathbf{w})$	$\mathbf{P}(\mathbf{w}^*)$
1 0 0 1 0 2	2 0 0 1 0 1
0 1 0 1 0 2	1 1 0 1 0 1
1 0 0 0 1 2	2 0 0 0 1 1
0 0 1 0 1 2	1 0 1 0 1 1

Some properties and applications of the Fourier transform

Properties:

- t-exact for the perverse t-structure and sends simples to simples.
- equivalence of categories
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Applications:

- character formula for quantum loop algebras uses Fourier transform on graded quiver varieties (Nakajima)
- monoidal categorification of certain cluster algebras (Nakajima)

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Theorem (Achar–K.–Matherne)

There is a bijection

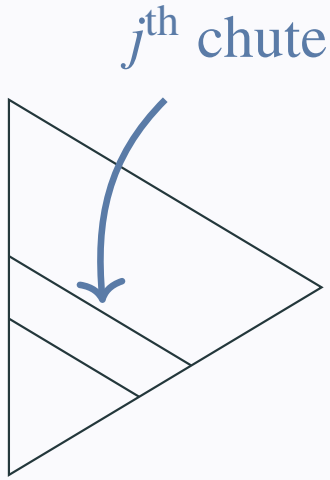
$$\mathbf{P}(\mathbf{w}) \xrightarrow{\mathsf{T}} \mathbf{P}(\mathbf{w}^*)$$

defined inductively by

$$\mathsf{T}\left(\begin{tikzcd}[column sep=small] \triangle(Y') \arrow[r, dash] & y_{1,n} \\ y_{n,1} \arrow[u, dash] \arrow[ur, dash, "\cdots"] & \end{tikzcd}\right) = \tau_n^{y_{1,n}} \tau_{n-1}^{y_{2,n-1}-y_{1,n}} \cdots \tau_1^{y_{n,1}-y_{n-1,2}} \left(\begin{tikzcd}[column sep=small] 0 \arrow[r, dash, "\ddots"] & 0 \\ \mathsf{T}(Y') \arrow[u, dash] & \end{tikzcd}\right)$$

where $\mathsf{T}(a) = a$.

Sliding at j



Define $\tau_j : \mathbf{P}(\mathbf{w}) \to \mathbf{P}(\mathbf{w} + \mathbf{e}_1 + \ldots + \mathbf{e}_j)$ by:

- Add 1 as far down the jth chute as possible, drawing an impassable vertical line there.
- Repeat for chutes $j-1, \ldots, 1$ not crossing lines.

Example of T

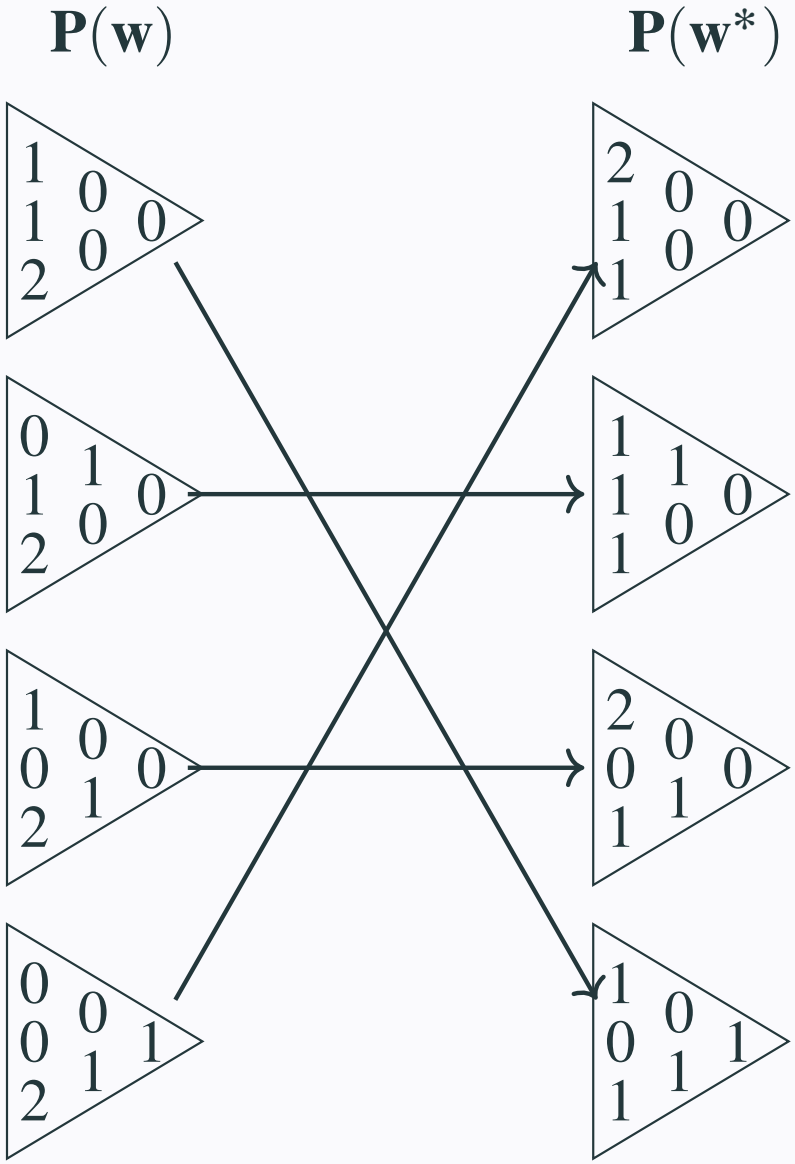
$$\begin{array}{lll} 1 & & \\ 0 & 0 & \\ 0 & & 0 \\ 2 & 1 & \end{array}$$

$$\mathsf{T}\left(\begin{array}{l} 1 \end{array}\right) = \begin{array}{l} 1 \end{array}$$

$$\mathsf{T}\left(\begin{array}{ll} 1 & \\ 0 & 0 \end{array}\right) = \begin{array}{ll} 0 & \\ 1 & 0 \end{array}$$

$$\mathsf{T}\left(\begin{array}{lll} 1 & & \\ 0 & 0 & \\ 0 & & 0 \\ 2 & 1 & \end{array}\right) = \tau_2\tau_1\left(\begin{array}{lll} 0 & & \\ 0 & 0 & \\ 0 & & 0 \\ 1 & 0 & \end{array}\right) = \begin{array}{lll} 2 & & \\ 0 & 0 & \\ 0 & & 0 \\ 1 & 1 & \end{array}$$

Running example



Main theorem

Theorem (Achar–K.–Matherne)

The bijection $\mathsf{T} : \mathbf{P}(\mathbf{w}) \to \mathbf{P}(\mathbf{w}^*)$ *determines* $\mathbb{T} : D^{\mathrm{b}}_{G(\mathbf{w})}(E(\mathbf{w})) \to D^{\mathrm{b}}_{G(\mathbf{w}^*)}(E(\mathbf{w}^*))$ *for simple perverse sheaves; that is,*

$$\mathbb{T}(\mathrm{IC}(\mathcal{O}_Y)) = \mathrm{IC}(\mathcal{O}_{\mathsf{T}(Y)}).$$

Outline of the proof

Geometric

Fourier transform

Combinatorial

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Multisegment duality

$B(w) \to B(w^*)$

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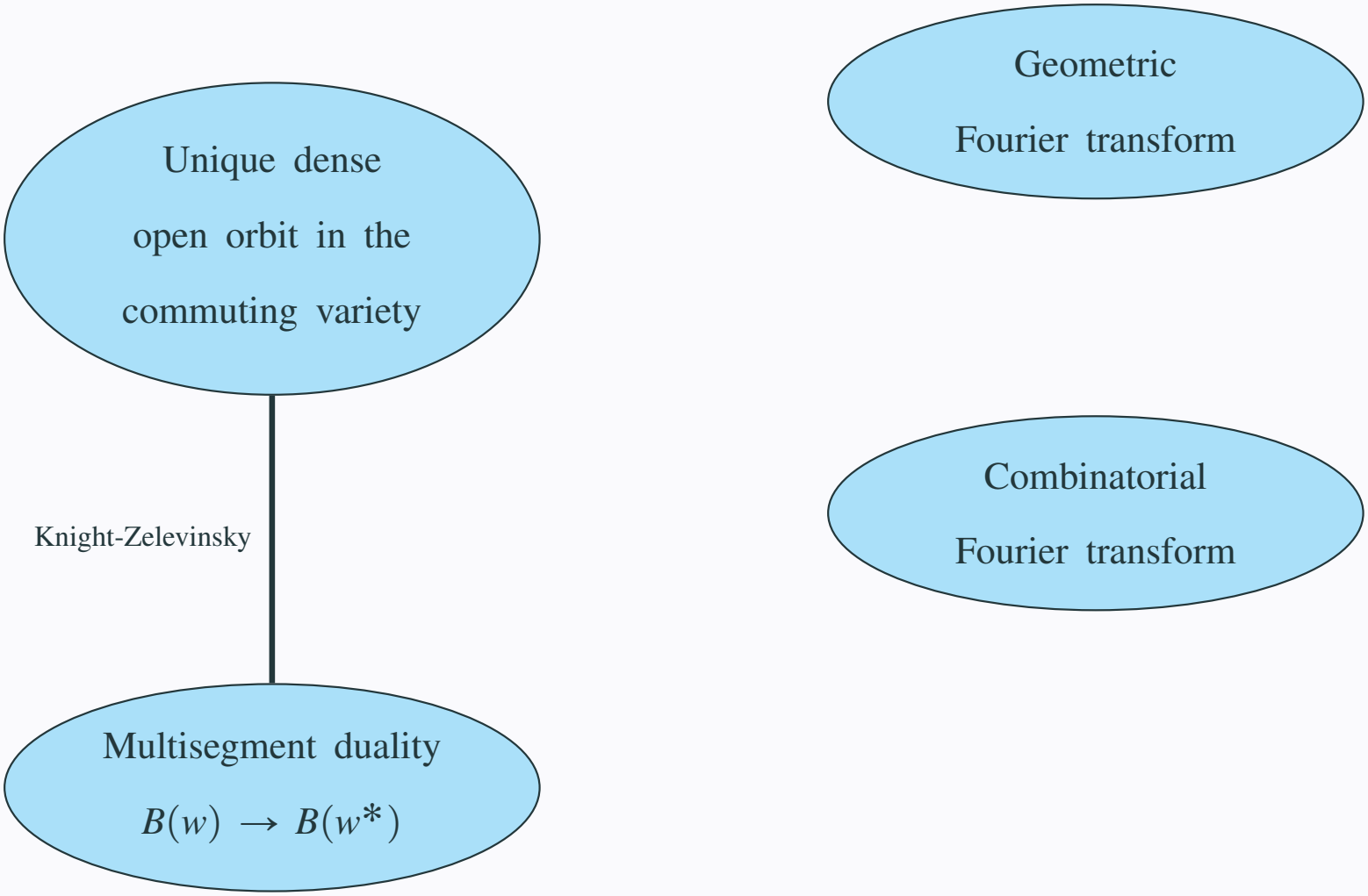

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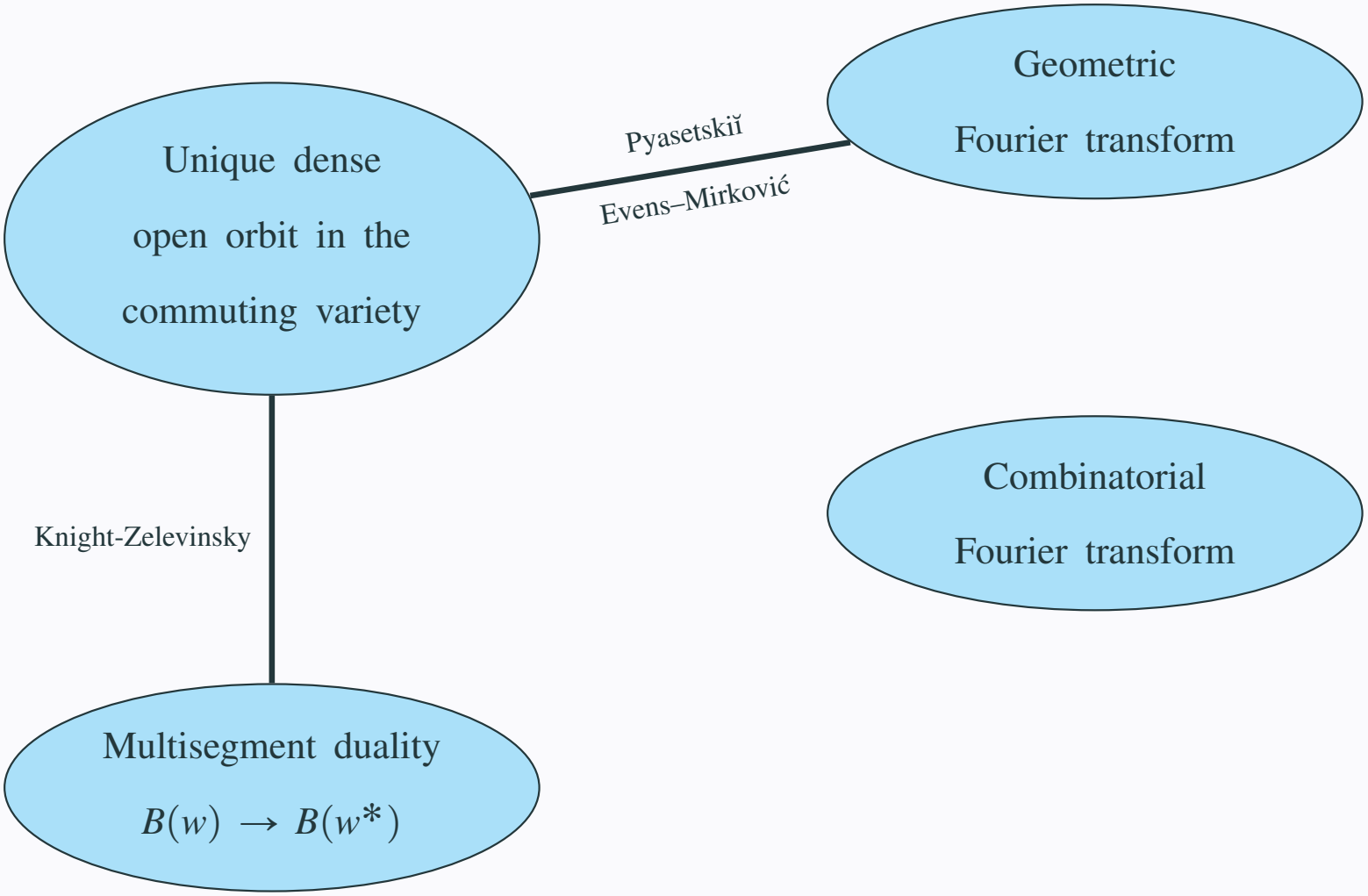

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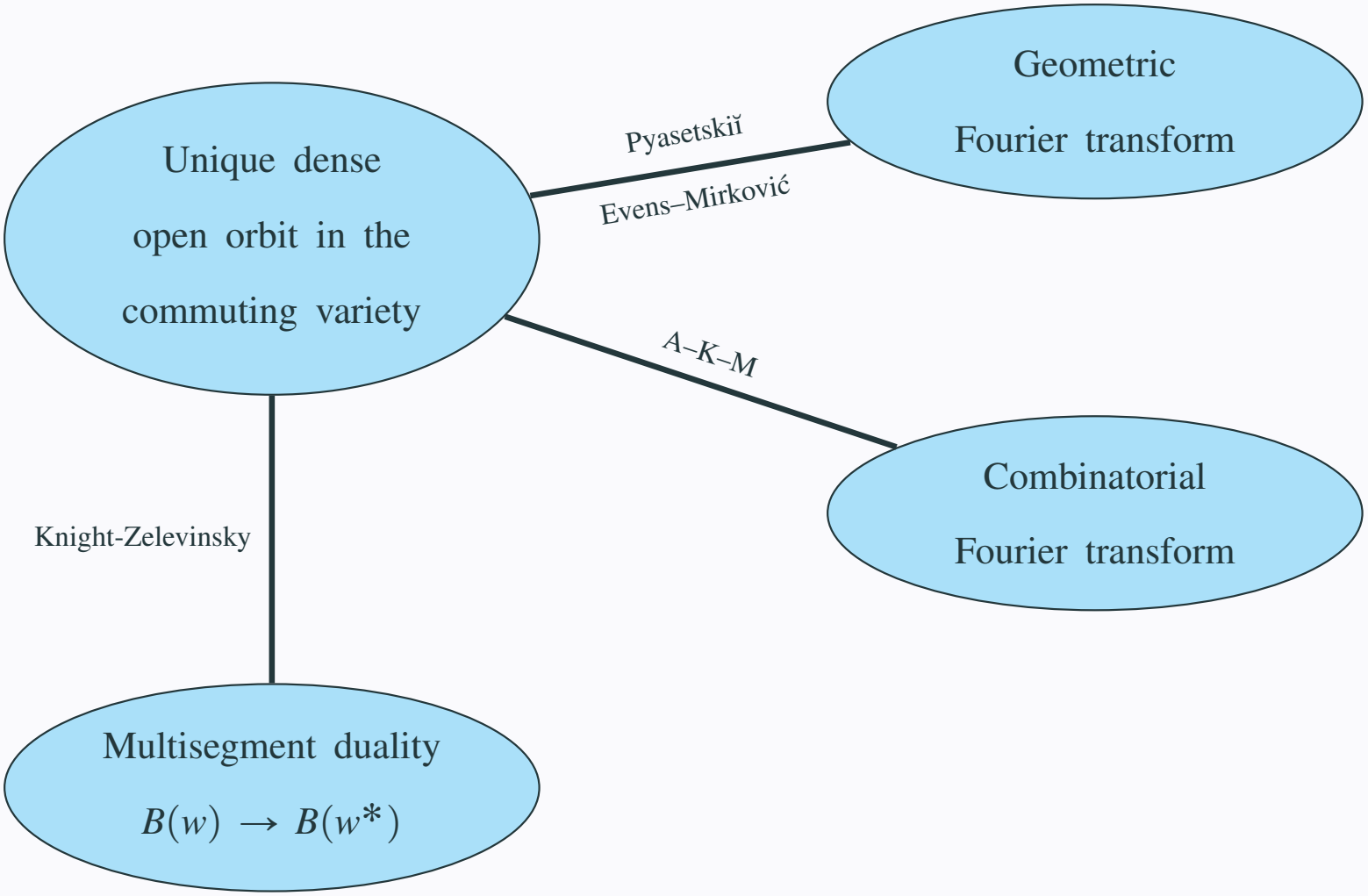

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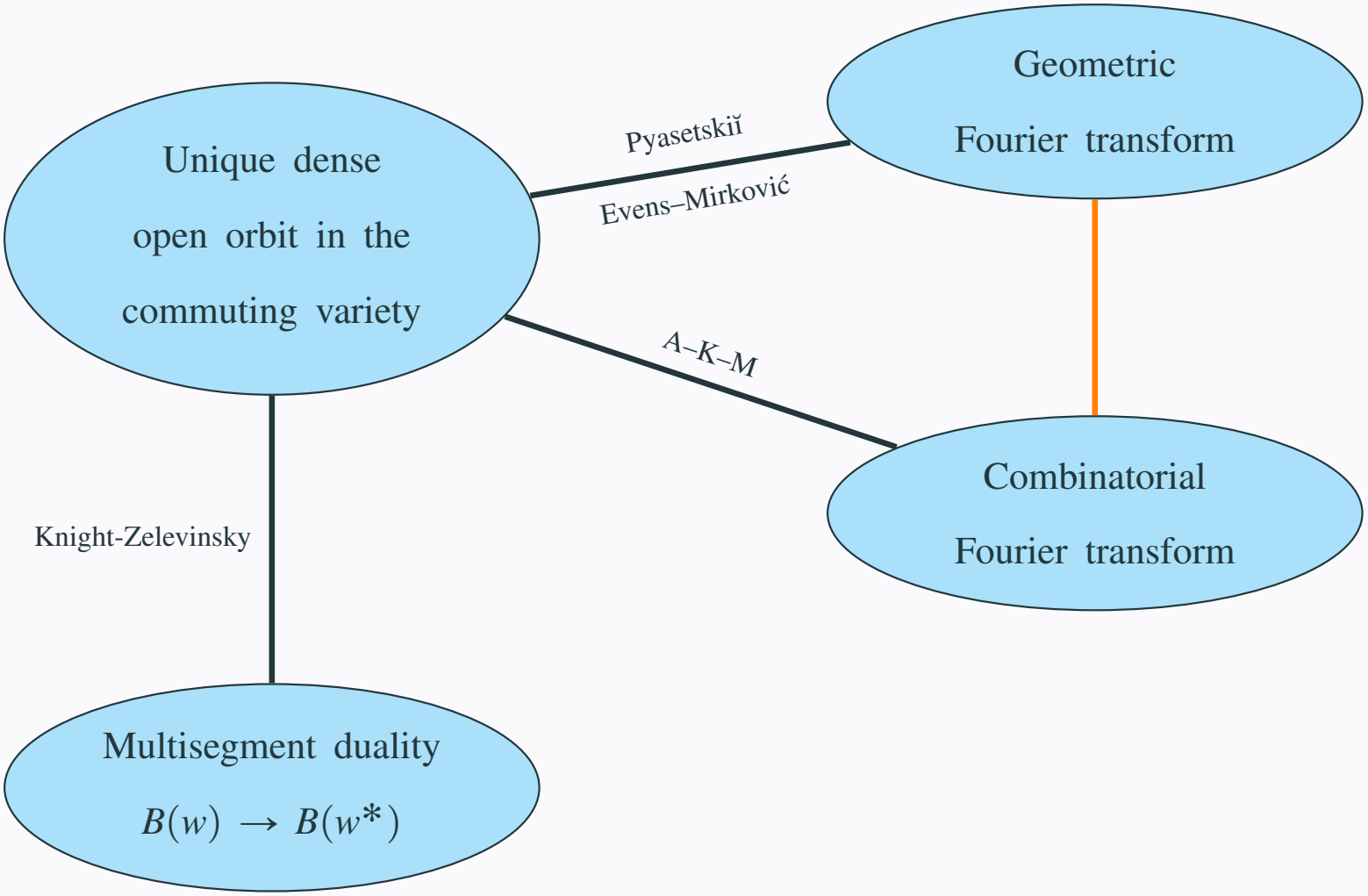

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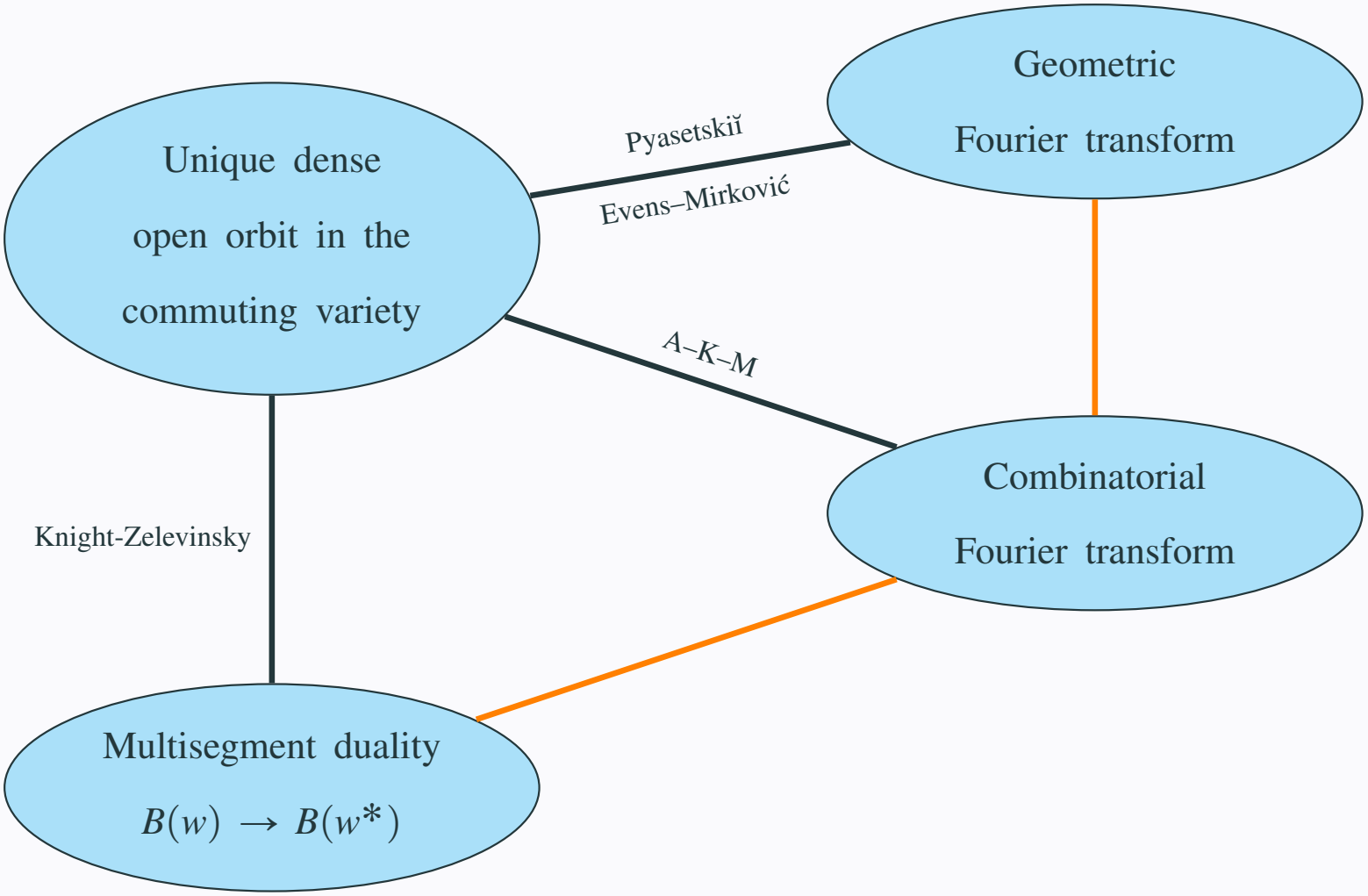

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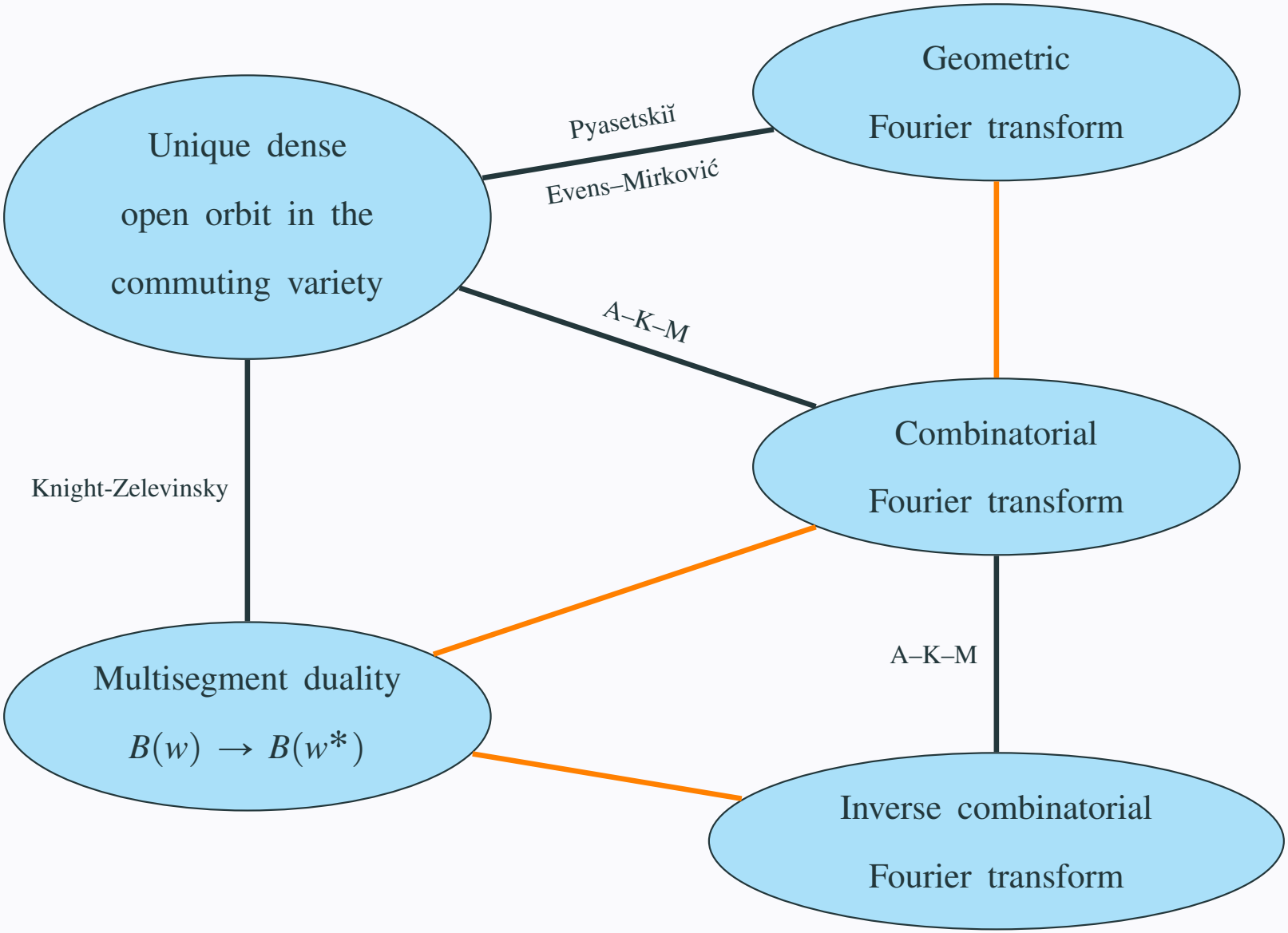

0 0 3 0 3 3
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2 1 3 1 2 0
2 2 3 0 1 1
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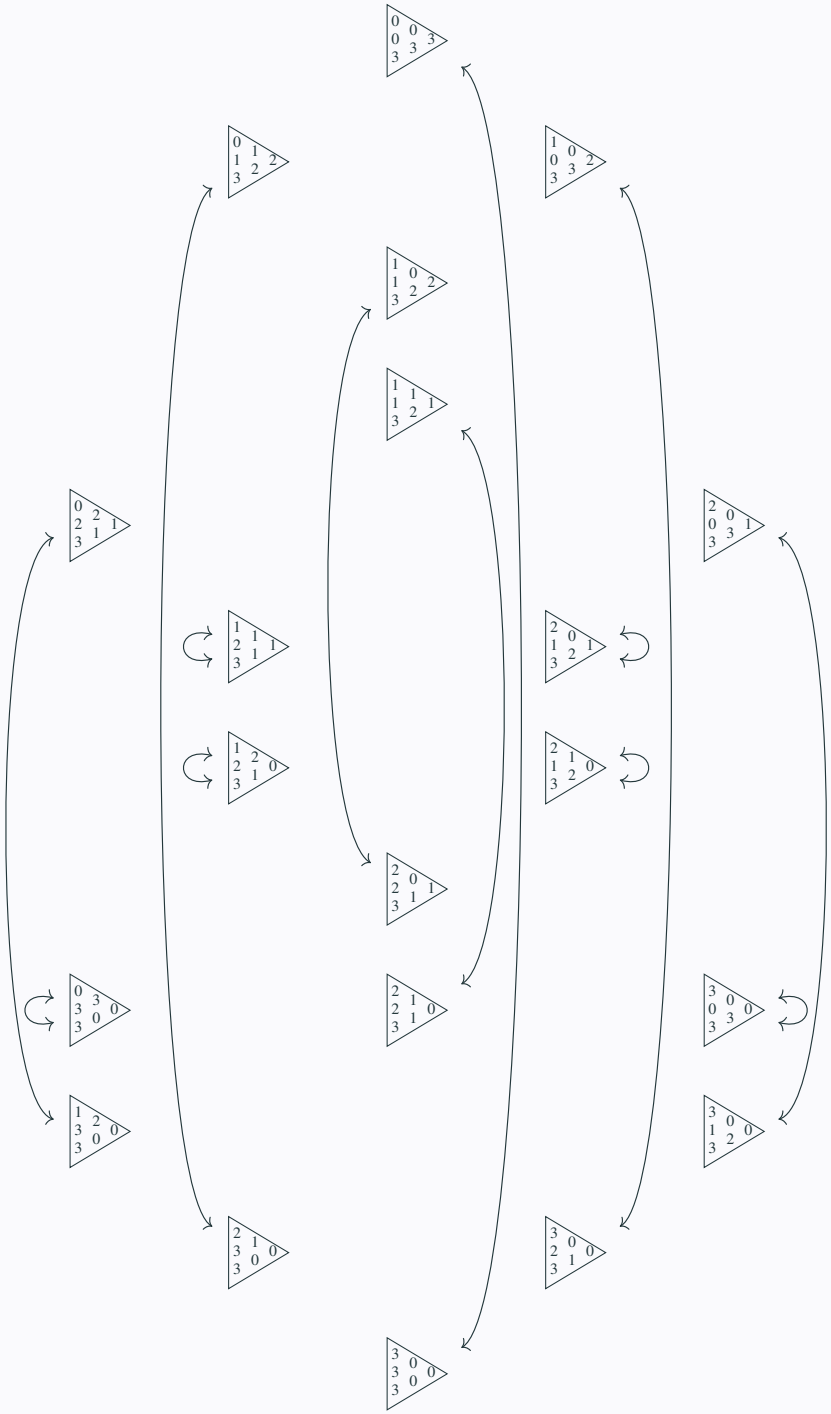

Thanks!