Feynman categories

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References

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Goals

Main Objective

Provide a *lingua universalis* for operations and relations in order to understand their structure.

Internal Applications

1. Realize universal constructions (e.g. free, push–forward, pull–back, plus construction, decorated).
2. Construct universal transforms. (e.g. bar,co–bar) and model category structure.
3. Distill universal operations in order to understand their origin (e.g. Lie brackets, BV operatos, Master equations).
4. Construct secondary objects, (e.g. Lie algebras, Hopf algebras).

Applications

Applications

- Find out information of objects with operations. E.g. Gromov-Witten invariants, String Topology, etc.
- Find out where certain algebra structures come from naturally: pre-Lie, BV, ...
- Find out origin and meaning of (quantum) master equations.
- Find background for certain types of Hopf algebras.
- Find formulation for TFTs.
- Transfer to other areas such as algebraic geometry, algebraic topology, mathematical physics, number theory, logic.

Plan

Warm up I

Operations and relations for Associative Algebras

- Data: An object A and a multiplication $\mu : A \otimes A \to A$
- An associativity equation $(ab)c = a(bc)$.
- Think of μ as a 2-linear map. Let $\circ_1$ and $\circ_2$ be substitution in the 1st resp. 2nd variable: The associativity becomes

$$\boxed{\mu \circ_1 \mu = \mu \circ_2 \mu : A \otimes A \otimes A \to A}.$$
$$\mu \circ_1 \mu(a, b, c) = \mu(\mu(a, b), c) = (ab)c$$
$$\mu \circ_2 \mu(a, b, c) = \mu(a, \mu(b, c)) = a(bc)$$

- We get n–linear functions by iterating μ: $a_1 \otimes \cdots \otimes a_n \to a_1 \dots a_n$.
- There is a permutation action $\tau\mu(a, b) = \mu \circ \tau(a, b) = ba$
- This give a permutation action on the iterates of μ. It is a free action there and there are $n!$ n–linear morphisms generated by μ and the transposition.

Warm up II

Categorical formulation for representations of a group G.

- $\underline{G}$ the category with one object $*$ and morphism set G.
- $f \circ g := fg$.
- This is associative ✓
- Inverses are an extra structure $\Rightarrow$ $\underline{G}$ is a groupoid.
- A representation is a functor ρ from $\underline{G}$ to $\mathcal{V}ect$.
- $\rho(*) = V$, $\rho(g) \in Aut(V)$
- Induction and restriction now are pull–back and push–forward (Lan) along functors $\underline{H} \to \underline{G}$.

Feynman categories

Data

1. $\mathcal{V}$ a groupoid
2. $\mathcal{F}$ a symmetric monoidal category
3. $\imath : \mathcal{V} \to \mathcal{F}$ a functor.

Notation

$\mathcal{V}^{\otimes}$ the free symmetric category on $\mathcal{V}$ (words in $\mathcal{V}$).

$$\begin{tikzcd} \mathcal{V} \arrow[rr, "\imath"] \arrow[dr, "\jmath"'] & & \mathcal{F} \\ & \mathcal{V}^{\otimes} \arrow[ur, "\imath^{\otimes}"'] & \end{tikzcd}$$

Feynman category

Definition

Such a triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ is called a Feynman category if

1. $\imath^{\otimes}$ induces an equivalence of symmetric monoidal categories between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
2. $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal categories between $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.
3. For any $* \in \mathcal{V}$, $(\mathcal{F} \downarrow *)$ is essentially small.

Basic consequences

1. $X \simeq \bigotimes_{v \in I} *_v$
2. $\phi : Y \to X$, $\phi \simeq \bigotimes_{v \in I} \phi_v$, $\phi_v : Y_v \to *_v$, $Y \simeq \bigotimes_{v \in I} Y_v$. The morphisms $\phi_v : Y \to *_v$ are called basic or one–comma generators.

“Representations” of Feynman categories: $\mathcal{O}ps$ and $\mathcal{M}ods$

Definition

Fix a symmetric cocomplete monoidal category $\mathcal{C}$, where colimits and tensor commute, and $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ a Feynman category.

- Consider the category of strong symmetric monoidal functors $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}} := Fun_{\otimes}(\mathcal{F}, \mathcal{C})$ which we will call $\mathcal{F}$–ops in $\mathcal{C}$
- $\mathcal{V}\text{-}\mathcal{M}ods_{\mathcal{C}} := Fun(\mathcal{V}, \mathcal{C})$ will be called $\mathcal{V}$-modules in $\mathcal{C}$ with elements being called a $\mathcal{V}$–mod in $\mathcal{C}$.

Trival *op*

Let $\mathcal{T} : \mathcal{F} \to \mathcal{C}$ be the functor that assigns $\mathbb{I} \in Obj(\mathcal{C})$ to any object, and which sends morphisms to the identity of the unit.

Remark

$\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{C}}$ is again a symmetric monoidal category.

Structure Theorems

Theorem

The forgetful functor $G : \mathcal{O}ps \to \mathcal{M}ods$ has a left adjoint F (free functor) and this adjunction is monadic. The endofunctor $\mathbb{T} = GF$ is a monad (triple) and $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$, algebras over the triple .

Theorem

Feynman categories form a 2–category and it has push–forwards f_* and pull–backs f^* for $\mathcal{O}ps$ and $\mathcal{M}ods$.

Remarks

Sometimes there is also a right adjoint $f_! = Ran_f$ which is "extension by zero" together with its adjoint $f^!$ will form part of a 6 functor formalism (see B. Ward).

Easy examples

$\mathcal{F} = \mathcal{V}^{\otimes}$, groupoid reps

$\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}} = \mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}} = Rep(\mathcal{V})$, that is groupoid representation.
Special case $\mathcal{V} = \underline{G} \rightsquigarrow$ Introduction.

Trivial $\mathcal{V}$

$\mathcal{V} = \underline{*}$, $\mathcal{V}^{\otimes} \simeq \underline{N}$ in the non–symmetric case and $\mathbb{S}$ in the symmetric case. Both categories have the natural numbers as objects and while $\underline{N}$ is discrete $Hom_{\mathbb{S}}(\underline{n}, \underline{n}) = \mathbb{S}_n$.
$\mathcal{V}$-$\mathcal{M}ods_{\mathcal{C}}$ are simply objects of $\mathcal{C}$.

Easy examples

$\mathcal{S}urj$, (commutative) Algebras

$\mathcal{F} = \mathcal{S}urj$ is finite sets with surjections. $Iso(sk(\mathcal{S}urj)) = \mathbb{S}$. $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ are commutative algebra objects in $\mathcal{C}$. Note; $\mathcal{O} \in \mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ then set $A = \mathcal{O}(\underline{1})$. As $\mathcal{O}$ is monoidal, $\mathcal{O}(\underline{n}) = A^{\otimes n}$, The surjection $\pi : \underline{2} \to \underline{1}$ gives the multiplication $\mu = \mathcal{O}(\pi) : A^{\otimes 2} \to A$. This is associative since $\pi \circ \pi \amalg id = \pi \circ id \amalg \pi = \pi_3 : \underline{3} \to \underline{1}$. The algebra is commutative, since $(12) \circ \pi = \pi$

Exercises

1. If once considers the non–symmetric analogue, one obtains ordered sets, with order preserving surjections and associative algebras.
2. What are the $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ for $\mathcal{F}in\mathcal{S}et$.

More examples with trivial $\mathcal{V}$

More examples of this type

1. Finite sets and injections.
2. $\Delta_+ S$ crossed simplicial group.

There is a non–symmetric monoidal version

Examples: Δ_+, also *"Simplices form an operad"*. Order preserving surjections/double base point preserving injections. Joyal duality.

$$Hom_{smCat}([n],[m]) = Hom_{*,*}([m+1],[n+1])$$

Enrichment sketch

Enrichment

There is a theory of enriched FCs. The axioms use Day convolution. Here (ii) is replaced by (ii'): the pull-back of presheaves $\imath^{\otimes\wedge}: [\mathcal{F}^{op}, Set] \to [\mathcal{V}^{\otimes op}, Set]$ *restricted to representable presheaves* is monoidal. This means

$$\imath^{\otimes\wedge} Hom_{\mathcal{F}}(\,\cdot\,, X\otimes Y) :=$$
$$Hom_{\mathcal{F}}(\imath^{\otimes}\,\cdot\,, X\otimes Y) = \imath^{\otimes\wedge} Hom_{\mathcal{F}}(\,\cdot\,, X) \circledast \imath^{\otimes\wedge} Hom_{\mathcal{F}}(\,\cdot\,, Y)$$
$$= \int^{Z,Z'} Hom_{\mathcal{F}}(\imath^{\otimes} Z, X) \times Hom_{\mathcal{F}}(\imath^{\otimes} Z', Y) \times Hom_{\mathcal{V}^{\otimes}}(\,\cdot\,, Z\otimes Z')$$

Enrichment, algebras (modules) There is a construction $\mathfrak{F}^+$ which gives nice enrichments.

Theorem/Definition [paraphrased]

$\mathfrak{F}^+$-$\mathcal{O}ps_{\mathcal{C}}$ are the enrichments of $\mathcal{F}$ (over $\mathcal{C}$). Given $\mathcal{O} \in \mathfrak{F}^+$-$\mathcal{O}ps_{\mathcal{C}}$ we denote by $\mathfrak{F}_{\mathcal{O}}$ the enrichment of $\mathfrak{F}$ by $\mathcal{O}$.

$$Hom_{\mathcal{F}_{\mathcal{O}}}(X, Y) = \bigoplus_{\phi \in Hom_{\mathcal{F}}(X,Y)} \mathcal{O}(\phi)$$

By definition the $\mathcal{F}_{\mathcal{O}}$-$\mathcal{O}ps_{\mathcal{E}}$ will be the algebras (modules) over $\mathcal{O}$.

Examples

$Tr^+ = \mathcal{S}urj$ (non-symmetric)/Modules

A an algebra then $\mathcal{T}r_A^+$ has objects $\underline{n}$ with $Hom(\underline{n}, \underline{n}) = A^{\otimes n}$ and hence we see that the $\mathcal{O}ps$ are just modules over A.

$\mathcal{S}urj^+ = \mathfrak{F}_{May}$/algebras over operads

$Hom_{\mathcal{S}urj_{\mathcal{O}}}(\underline{n}, \underline{1}) = \mathcal{O}(n)$. Composition of morphisms $\underline{n} \xrightarrow{f} \underline{k} \to \underline{1}$

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n)$$

where $n_i = |f^{-1}(i)|$.
So $\mathcal{O}ps$ are algebras over the operad $\mathcal{O}$.

Why Feynman? Graphical Feynman categories.

Physics (connected case)

Objects of $\mathcal{V}$ are the vertices of the theory. The morphims of $\mathcal{F}$ "are" the possible Feynman graphs. Both can be read off the Lagrangian or actions.
The source of a morphisms ϕ_Γ "is" the set of vertices $V(\Gamma)$ and the target of a basic morphism is the external leg structure $\Gamma/E(\Gamma)$.
The terms in the S matrix corresponding to the external leg structure $*$ is $(\mathcal{F} \downarrow *_v)$.

Math

Basic graphs, full subcategory of Borisov-Manin category of graphs whose objects are aggregates of corollas (no edges). The morphisms have an underlying graph, the ghost graph.

Examples

Roughly (in the connected case and up to isomorphism)

The source of a morphism are the vertices of the ghost graph $\mathbb{\Gamma}$ and the target is the vertex obtained from $\mathbb{\Gamma}$ obtained by contracting all edges. If $\mathbb{\Gamma}$ is not connected, one also needs to merge vertices according to ϕ_V.

Composition corresponds to insertion of ghost graphs into vertices.

$$\begin{tikzcd} X \arrow[r, "\phi_2"] \arrow[rr, bend right, "\phi_0"'] & Y \arrow[r, "\phi_1"] & * \end{tikzcd}$$

up to isomorphisms (if $\mathbb{\Gamma}_0, \mathbb{\Gamma}_1$ are connected) corresponds to inserting $\mathbb{\Gamma}_v$ into $*_v$ of $\mathbb{\Gamma}_1$ to obtain $\mathbb{\Gamma}_0$.

$$\begin{tikzcd} \amalg_v \amalg_{w \in V_v} *_w \arrow[r, "\amalg_v \mathbb{\Gamma}_v"] \arrow[rr, bend right, "\mathbb{\Gamma}_0"'] & \amalg_v *_v \arrow[r, "\mathbb{\Gamma}_1"] & * \end{tikzcd}$$

Examples based on $\mathfrak{G}$: morphisms have underlying graphs

$\mathfrak{F}$	Feynman category for	condition on graphs additional decoration
$\mathfrak{O}$	operads	rooted trees
$\mathfrak{O}_{mult}$	operads with mult.	b/w rooted trees.
$\mathfrak{C}$	cyclic operads	trees
$\mathfrak{G}$	unmarked nc modular operads	graphs
$\mathfrak{G}^{ctd}$	unmarked modular operads	connected graphs
$\mathfrak{M}$	modular operads	connected + genus marking
$\mathfrak{M}^{nc,}$	nc modular operads	genus marking
$\mathfrak{D}$	dioperads	connected directed graphs w/o directed loops or parallel edges
$\mathfrak{P}$	PROPs	directed graphs w/o directed loops
$\mathfrak{P}^{ctd}$	properads	connected directed graphs w/o directed loops
$\mathfrak{D}^{\circlearrowright}$	wheeled dioperads	directed graphs w/o parallel edges
$\mathfrak{P}^{\circlearrowright,ctd}$	wheeled properads	connected directed graphs
$\mathfrak{P}^{\circlearrowright}$	wheeled props	directed graphs

Table: List of Feynman categories with conditions and decorations on the graphs, yielding the zoo of examples

Examples on $\mathfrak{G}$ with extra decorations

Decoration and restriction allows to generate the whole zoo and even new species

$\mathfrak{F}_{dec\mathcal{O}}$	Feynman category for	decorating $\mathcal{O}$	restriction
$\mathfrak{F}^{dir}$	directed version	$\mathbb{Z}/2\mathbb{Z}$ set	edges contain one input and one output flag
$\mathfrak{F}^{rooted}$	root	$\mathbb{Z}/2\mathbb{Z}$ set	vertices have one output flag.
$\mathfrak{F}^{genus}$	genus marked	$\mathbb{N}$	
$\mathfrak{F}^{c-col}$	colored version	c set	edges contain flags of same color
$\mathfrak{O}^{\neg\Sigma}$	non-Sigma-operads	*Ass*	
$\mathfrak{C}^{\neg\Sigma}$	non-Sigma-cyclic operads	*CycAss*	
$\mathfrak{M}^{\neg\Sigma}$	non–Signa-modular	*ModAss*	
$\mathfrak{C}^{dihed}$	dihedral	*Dihed*	
$\mathfrak{M}^{dihed}$	dihedral modular	*ModDihed*	

Table: List of decorates Feynman categories with decorating $\mathcal{O}$ and possible restriction. $\mathfrak{F}$ stands for an example based on $\mathfrak{G}$ in the list.

Constructions yielding Feynman categories

A partial list

1. $+$ construction: Twisted modular operads, twisted versions of any of the previous structures. Quotient gives $\mathfrak{F}^{hyp}$.
2. $\mathcal{F}_{dec\mathcal{O}}$: non–Sigma and dihedral versions.It also yields all graph decorations.
3. free constructions $\mathfrak{F}^{\boxtimes}$, s.t. $\mathfrak{F}^{\boxtimes}$-$\mathcal{O}ps_{\mathcal{C}} = Fun(\mathcal{F}, \mathcal{C})$. Used for the simplicial category, crossed simplicial groups and FI–algebras.
4. Non–connected construction $\mathfrak{F}^{nc}$, whose $\mathcal{F}^{nc}$-$\mathcal{O}ps$ are equivalent to lax monoidal functors of $\mathcal{F}$.
5. The Feynman category of universal operations on $\mathfrak{F}$–$\mathcal{O}ps$.
6. Cobar/bar, Feynman transforms in analogy to algebras and (modular) operads.
7. W–construction.

Results

Theorem

The Feynman transform of a non-negatively graded dg $\mathcal{F}$-op is cofibrant.
The double Feynman transform of a non-negatively graded dg $\mathcal{F}$-op in a quadratic Feynman category is a cofibrant replacement.

Theorem

Let $\mathfrak{F}$ be a simple Feynman category and let $\mathcal{P} \in \mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$ be ρ-cofibrant. Then $W(\mathcal{P})$ is a cofibrant replacement for $\mathcal{P}$ with respect to the above model structure on $\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$.

Here "simple" is a technical condition satisfied by all graph examples.

$\mathfrak{F}_{dec\mathcal{O}}$ joint w/ Jason Lucas

Theorem

Given an $\mathcal{O} \in \mathcal{F}$–$\mathcal{O}ps$, then there is a Feynman category $\mathcal{F}_{dec\mathcal{O}}$ which is indexed over $\mathcal{F}$.

- *It objects are pairs $(X, dec \in \mathcal{O}(X))$*
- *$Hom_{\mathcal{F}_{dec\mathcal{O}}}((X, dec), (X', dec'))$ is the set of $\phi : X \to X'$, s.t. $\mathcal{O}(\phi)(dec) = dec'$.*

(This construction works a priori for Cartesian $\mathcal{C}$, but with modifications it also works for the non–Cartesian case.)

Example

$\mathfrak{F} = \mathfrak{C}$, $\mathcal{O} = CycAss$, $CycAss(*_S) = \{\text{cyclic orders} \prec \text{on } S\}$. New basic objects of $\mathfrak{C}_{decCycAss}$ are planar corollas $*_{S,\prec}$. Morphisms "are planar trees".

Results

Theorem

(1)

$$\begin{tikzcd}
\mathfrak{F}_{dec\mathcal{O}} \arrow[r, "f^{\mathcal{O}}"] \arrow[d, "forget"'] & \mathfrak{F}'_{dec\ f_*(\mathcal{O})} \arrow[d, "forget'"] \\
\mathfrak{F} \arrow[r, "f"] & \mathfrak{F}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{F}_{dec\mathcal{O}} \arrow[r, "\sigma_{dec}"] \arrow[d, "f^{\mathcal{O}}"'] & \mathfrak{F}_{dec\mathcal{P}} \arrow[d, "f^{\mathcal{P}}"] \\
\mathfrak{F}'_{decf_*(\mathcal{O})} \arrow[r, "\sigma'_{dec}"] & \mathfrak{F}'_{decf_*(\mathcal{P})}
\end{tikzcd}$$

The squares above commute squares and are natural in $\mathcal{O}$. We get the induced diagram of adjoint functors.

(2)

$$\begin{tikzcd}
\mathcal{F}_{dec\mathcal{O}}\text{-}\mathcal{O}ps \arrow[r, shift left, "f^{\mathcal{O}}_*"] \arrow[d, shift right, bend right, "forget_*"'] & \mathcal{F}'_{dec\ f_*(\mathcal{O})}\text{-}\mathcal{O}ps \arrow[l, shift left, "f^{\mathcal{O}*}"] \arrow[d, shift left, bend left, "forget'_*"] \\
\mathcal{F}\text{-}\mathcal{O}ps \arrow[r, shift left, "f_*"] \arrow[u, shift right, bend right, "forget^*"'] & \mathcal{F}'\text{-}\mathcal{O}ps \arrow[l, shift left, "f^*"] \arrow[u, shift left, bend left, "forget'^*"]
\end{tikzcd}$$

More $\mathcal{F}_{dec\mathcal{O}}$

Theorem

If $\mathcal{T}$ is a terminal object for $\mathcal{F}$-$\mathcal{O}ps$ and forget : $\mathcal{F}_{dec\mathcal{O}} \to \mathcal{F}$ is the forgetful functor, then forget$^*(\mathcal{T})$ is a terminal object for $\mathcal{F}_{dec\mathcal{O}}$-$\mathcal{O}ps$. We have that forget$_*$forget$^*(\mathcal{T}) = \mathcal{O}$.

Definition

We call a morphism of Feynman categories $i : \mathfrak{F} \to \mathfrak{F}'$ a minimal extension over $\mathcal{C}$ if $\mathfrak{F}$-$\mathcal{O}ps_{\mathcal{C}}$ has a a terminal/trivial functor $\mathcal{T}$ and $i_*\mathcal{T}$ is a terminal/trivial functor in $\mathfrak{F}'$-$\mathcal{O}ps_{\mathcal{C}}$.

Proposition

If $f : \mathfrak{F} \to \mathfrak{F}'$ is a minimal extension over $\mathcal{C}$, then $f^{\mathcal{O}} : \mathfrak{F}_{dec\mathcal{O}} \to \mathfrak{F}'_{decf_*(\mathcal{O})}$ is as well.

Factorization

Theorem (w/ C. Berger)

Any morphisms of Feynman $f : \mathfrak{F} \to \mathfrak{F}'$ categories factors and a minimal extension followed by a decoration cover.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{F}'_{dec\, f_*(\mathcal{T})} \arrow[d] \\ & \mathcal{F}' \end{tikzcd}$$

Example

Bootstrap

(3)
$$
\begin{tikzcd}
\mathfrak{C}_{dec\ CycAss} = \mathfrak{C}^{\neg\Sigma} \arrow[r, "i^{CycAss}"] \arrow[d, "forget"'] & \mathfrak{M}_{dec\ i_*(CycAss)} = \mathfrak{M}^{\neg\Sigma} \arrow[d, "forget"] \\
\mathfrak{C} \arrow[r, "i"] \arrow[dr, "j"] & \mathfrak{M} = \mathfrak{G}^{ctd}_{j_*(\mathcal{T})} \arrow[d, "forget"] \\
& \mathfrak{G}^{ctd}
\end{tikzcd}
$$

1. $\mathfrak{C}$-$\mathcal{O}ps$ are cyclic operads. Basic graphs are trees.
2. $\mathfrak{G}^{ctd}$: Basic graphs are connected graphs.
3. $j_*(\mathcal{T})(*_S) = \amalg_{g\in\mathbb{N}} *$ hence elements of $\mathcal{V}$ for $\mathfrak{M}$ are of the form $*_{g,S}$ they can be thought of an oriented surface of genus g and S boundaries.

Example

Bootstrap

(4)
$$\begin{tikzcd}
\mathfrak{C}_{dec\ CycAss} = \mathfrak{C}^{\neg\Sigma} \arrow[r, "i^{CycAss}"] \arrow[d, "forget"'] & \mathfrak{M}_{dec\ i_*(CycAss)} = \mathfrak{M}^{\neg\Sigma} \arrow[d, "forget"] \\
\mathfrak{C} \arrow[r, "i"] \arrow[dr, "j"] & \mathfrak{M} = \mathfrak{G}^{ctd}_{j_*(\mathcal{T})} \arrow[d, "forget"] \\
& \mathfrak{G}^{ctd}
\end{tikzcd}$$

4. $\mathfrak{M}^{\neg\Sigma}$ are non–sigma modular operads (Markl, K-Penner). Elements of $\mathcal{V}$ are $*_{g,s,S_1,\dots,S_b}$ where each S_i has a cyclic order. These can be thought of as oriented surfaces with genus g, s internal marked points, b boundaries where each boundary i has marked points labelled by S_i in the given cyclic order.

Hopf algebras

Basic structures

Assume $\mathcal{F}$ is decomposition finite. Consider $\mathcal{B} = Hom(Mor(\mathcal{F}), \mathbb{Z})$. Let μ be the tensor product with unit $id_{\mathbb{I}}$.

$$\Delta(\phi) = \sum_{(\phi_0,\phi_1):\phi=\phi_1\circ\phi_0} \phi_0 \otimes \phi_1$$

and $\epsilon(\phi) = 1$ if $\phi = id_X$ and 0 else.

Theorem (Galvez-Carrillo, K , Tonks)

$\mathcal{B}$ together with the structures above is a bi–algebra. Under certain mild assumptions, a canonical quotient is a Hopf algebra

$\mathcal{S}urj_{\mathcal{O}}$

$k[Mor(\mathcal{S}urj_{\mathcal{O}})]$ are the free cooperad with multiplication on a cooperad

$\check{\mathcal{O}}^{nc}(n) = \bigoplus_k \bigoplus_{(n_1,\dots,n_k):\sum n_i=n} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k)$
Multiplication given by $\mu = \otimes$.

Hopf algebras/(co)operads/Feynman category

H_{Gont}	$Inj_{*,*} = Surj^*$	$\mathfrak{F}_{Surj}$
H_{CK}	leaf labelled trees	$\mathfrak{F}_{Surj,\mathcal{O}}$
$H_{CK,graphs}$	graphs	$\mathfrak{F}_{graphs}$
H_{Baues}	$Inj^{gr}_{*,*}$	$\mathfrak{F}_{Surj,odd}$

Example

Examples

In this fashion, we can reproduce Connes–Kreimer's Hopf algebra, the Hopf algebras of Goncharov and a Hopf algebra of Baues that he defined for double loop spaces. This is a non–commutative graded version. There is a three-fold hierarchy. A non-commutative version, a commutative version and an "amputated" version.

Extension

Extension to not necessarily free cooperads with multiplication. $\Delta = (id \otimes \mu^{\otimes n}) \circ \check{\gamma}$. Filtrations instead of grading. Developable and deformation of associated graded.

W-construction

Input: Cubical Feynman categories in a nutshell

- Generators and relations for basic morphisms.
- Additive length function $l(\phi)$, $l(\phi)=0$ equivalent to ϕ is iso.
- Quadratic relations and every morphism of length n has precisely $n!$ decompositions into morphisms of length 1 up to isomorphisms.
- Ex: $\phi_{e_1}\circ\phi_{e_2}=\phi_{e_2'}\circ\phi_{e_1'}$, commutative square for edge contractions.

Definition

Let $\mathcal{P}\in\mathcal{F}\text{-}\mathcal{O}ps_{\mathcal{T}op}$. For $Y\in ob(\mathcal{F})$ we define

$$W(\mathcal{P})(Y):=colim_{w(\mathfrak{F},Y)}\mathcal{P}\circ s(-)$$

Details

The category $w(\mathfrak{F}, Y)$, for $Y \in \mathcal{F}$ Objects:

Objects are the set $\coprod_n C_n(X,Y) \times [0,1]^n$, where $C_n(X,Y)$ are chains of morphisms from X to Y with n degree ≥ 1 maps modulo contraction of isomorphisms.
An object in $w(\mathfrak{F}, Y)$ will be represented (uniquely up to contraction of isomorphisms) by a diagram

$$X \xrightarrow[f_1]{t_1} X_1 \xrightarrow[f_2]{t_2} X_2 \to \cdots \to X_{n-1} \xrightarrow[f_n]{t_n} Y$$

where each morphism is of positive degree and where $t_1, \ldots, t_n$ represents a point in $[0,1]^n$. These numbers will be called weights. Note that in this labeling scheme isomorphisms are always unweighted.

Setup: quadratic Feynman category $\mathfrak{F}$

The category $w(\mathfrak{F}, Y)$, for $Y \in \mathcal{F}$ Morphisms:

1. Levelwise commuting isomorphisms which fix Y, i.e.:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\cong"] & X_1 \arrow[r] \arrow[d, "\cong"] & X_2 \arrow[r] \arrow[d, "\cong"] & \dots \arrow[r] & X_n \arrow[r] \arrow[d, "\cong"] & Y \\
X' \arrow[r] & X'_1 \arrow[r] & X'_2 \arrow[r] & \dots \arrow[r] & X'_n \arrow[ur] &
\end{tikzcd}$$

2. Simultaneous $\mathbb{S}_n$ action.
3. Truncation of 0 weights: morphisms of the form $(X_1 \xrightarrow{0} X_2 \to \cdots \to Y) \mapsto (X_2 \to \cdots \to Y)$.
4. Decomposition of identical weights: morphisms of the form $(\cdots \to X_i \xrightarrow{t} X_{i+2} \to \dots) \mapsto (\cdots \to X_i \xrightarrow{t} X_{i+1} \xrightarrow{t} X_{i+2} \to \dots)$ for each (composition preserving) decomposition of a morphism of degree ≥ 2 into two morphisms each of degree ≥ 1.

Cubical decomposition of associahedra

$W(Ass)$

The associative operad $Ass(n) = regular(\mathbb{S}_n)$. $W(Ass)(n)$ is a cubical decomposition of the associahedron.

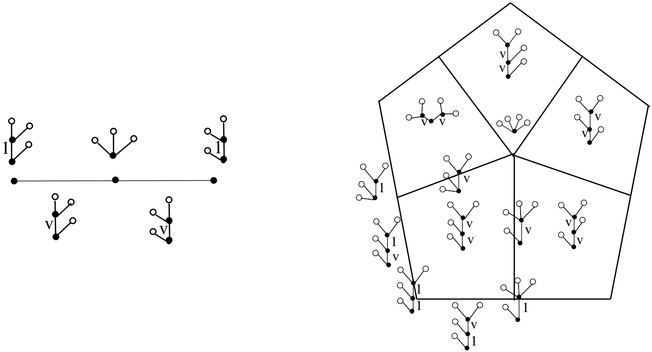

Figure: The cubical decomposition for K_3 and K_4, v indicates a variable height.

Models for moduli spaces and push–forwards

The square revisited

$$\begin{tikzcd} \mathcal{F}_{dec\ CycAss} = \mathfrak{C}^{\neg\Sigma} \arrow[r, "i^{CycAss}"] \arrow[d, "forget"'] & \mathfrak{M}_{dec\ i_*(CycAss)} = \mathfrak{M}^{\neg\Sigma} \arrow[d, "forget"] \\ \mathfrak{C} \arrow[r, "i"] & \mathfrak{M} \end{tikzcd}$$

Work with C. Berger

1. $Wi_*(CycAss) = (*_{g,n}) = Cone(\bar{M}^{K/P}_{g,n}) \supset \bar{M}^{K/P}_{g,n} \supset M_{g,n}$, metric almost ribbon graphs (emtpy graph is allowed).
2. $i_*^{cycAss}W\mathcal{T})(*_{g,s,S_1\amalg\cdots\amalg S_b}) \simeq B\Gamma_{g,s,S_1\amalg\cdots\amalg S_b}$. This is a generalization of Igusa's theorem $B\Gamma_{g,n} = Nerve(IgusaCat)$
3. $FT(i_*(CycAss))(*_{g,n}) = CC_*(\bar{M}^{K/P}_{g,n})$.

Cutkosky/Outer space, w/ C. Berger

The cube complex $j_*(W(CycAss))(*_S)$

Is the complex whose cubical cells are indexed by pairs (Γ, τ), where

- Γ is a graph with S–labelled tails and τ is a spanning forrest.
- The cell has dimension $|E(\tau)|$
- the differential ∂_e^- contracts the edge
- ∂_e^+, removes the edge from the spanning forrest.

Pictures for an algebra restriction

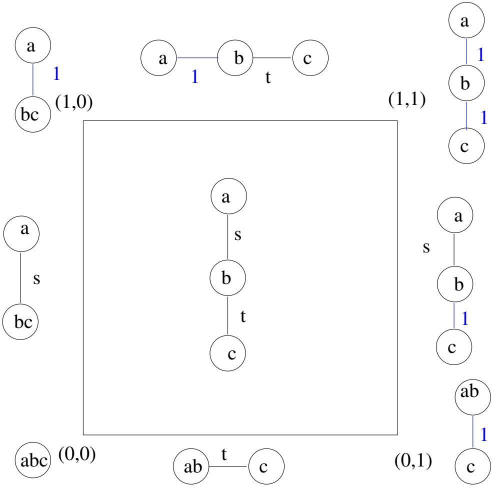

Figure: The cubical structure in the case of $n = 3$. One can think of the edges marked by 1 as cut.

Other interpretations of the same picture

Remark

The cubical structure also becomes apparent if we interpret $[n] = 0 \to 1 \to 2 \to \cdots \to n$ as the simplex.

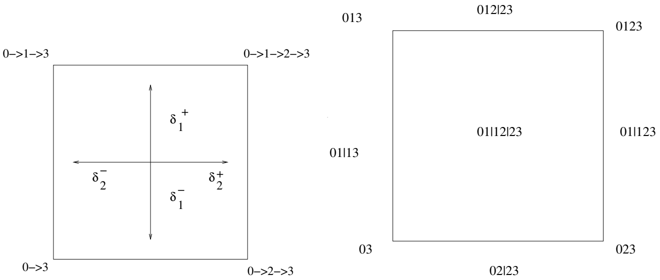

Figure: Two other renderings of the same square. Note: $0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$

Next steps

- Formalize the dual pictures of primitive elements and $+$ construction as well as universal operations and PBW. (Idea: special properties of $\mathcal{H}_{CK}$).
- Connect to Rota–Baxer, Dynkin-operators, B^+-operators (we can do this part) etc.
- Formalize string topology operations.
- Connect to quiver theories and to stability conditions. Wall crossing corresponds to contracting and expanding an edge.
- More quadratic ...
- ...

The end

Thank you!