d-abelian quotients of $d+2$-angulated categories

Joint work with Peter Jørgensen

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Motivation

Tilting theory is useful when dealing with abelian and triangulated categories.

It would be really neat to be able to use it with d-abelian and $d+2$-angulated categories

This is one step towards that...

Setup

- $k = \overline{k}$ is a field
- All categories are additive and k-linear.
- d is a positive integer (if $d = 1$ we get the classical case).

d-cluster-tilting subcategories

Definition (Iyama 2010)

Let $\mathscr{C}$ be an abelian or triangulated category. Let $\mathscr{X} \subseteq \mathscr{C}$ be a full subcategory.

- $\mathscr{X}$ is *d-rigid* if $\mathrm{Ext}^i_{\mathscr{C}}(\mathscr{X}, \mathscr{X}) = 0$ for $1 \leqslant i \leqslant d-1$
- $\mathscr{X}$ is *weakly d-cluster tilting* if

$$\begin{aligned}\mathscr{X} &= \{C \in \mathscr{C} \mid \mathrm{Ext}^i_{\mathscr{C}}(C, \mathscr{X}) = 0 \text{ for } 1 \leqslant i \leqslant d-1\} \\ &= \{C \in \mathscr{C} \mid \mathrm{Ext}^i_{\mathscr{C}}(\mathscr{X}, C) = 0 \text{ for } 1 \leqslant i \leqslant d-1\}.\end{aligned}$$

- $\mathscr{X}$ is *d-cluster tilting* if it is weakly d-cluster tilting and functorially finite in $\mathscr{C}$.

Apply the same adjective to an object T if the condition holds for $\mathscr{X} = \mathrm{Add}\, T$

d-abelian categories

Abelian categories	d-abelian categories
Short exact sequences	d-exact sequences
$0 \to X \to Y \to Z \to 0$	$0 \to X \to Y_d \to \cdots \to Y_1 \to Z \to 0$
Kernels and cokernels	d-kernels and d-cokernels
Projective resolutions	Projective resolutions of length at least d

Theorem (Jasso 2016)

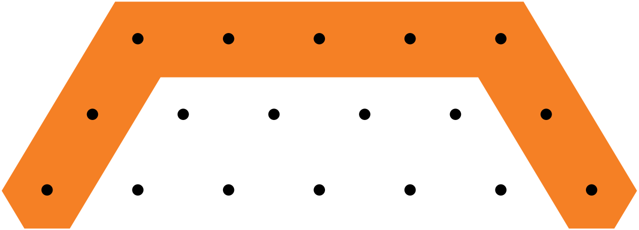
Let $\mathscr{A}$ be abelian and let $\mathscr{X} \subseteq \mathscr{A}$ be d-cluster-tilting. Then $\mathscr{X}$ is d-abelian.

Example

Theorem (Vaso 2016)

Let $\Gamma = k\mathbb{A}_n/\langle \text{paths of length } l\rangle$ with $l = 2$ or $n \equiv 1 \pmod{l}$.
Let $\mathscr{X} \subseteq \operatorname{mod}\Gamma$ be all projective and injective modules in $\operatorname{mod}\Gamma$.
Then $\mathscr{X}$ is d-clustertilting and thus d-abelian.

$n = 7$, $l = 3$: $d = 4$



$d+2$-angulated categories

Definition due to [Geiss, Keller, Oppermann, 2013]
Suppose $\mathscr{C}$ is k-linear and Krull-Schmidt.
Let Σ^d be an autoequivalence on $\mathscr{C}$, called a d-suspension.
Suppose we can define a collection of $d+2$-angles,

$$X_{d+2} \to X_{d+1} \to \cdots \to X_1 \to \Sigma^d X_{d+2},$$

that act pretty much like the triangles in a triangulated category
(Don't make me give you the axioms...)
Then we call $\mathscr{C}$ a $d+2$-angulated category.

Theorem (Geiss, Keller, Oppermann 2013)

Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{X} \subseteq \mathscr{C}$ be a d-cluster-tilting subcategory.
Then $\mathscr{X}$ is a $d+2$-angulated category.

Example(s)

Theorem (Oppermann, Thomas 2012)

Suppose Γ is d-representation-finite. Let $\mathscr{X} \subseteq \operatorname{mod}\Gamma$ be d-cluster-tilting. Then

$$\mathscr{Y} = \{\mathscr{X}[nd] \mid n \in \mathbb{Z}\} \subseteq D^b(\operatorname{mod}\Gamma)$$

is a d-cluster-tilting subcategory and thus $d+2$-angulated. The d-suspension functor is $[d]$.

In the case of our previous example we get something that's at least easy to calculate:

$$\begin{tikzcd}
\cdots \arrow[r] & \bullet \arrow[r] \arrow[rr, bend left] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[r, bend left] & \bullet \arrow[r] \arrow[rr, bend left] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[r, bend left] & \bullet \arrow[r] \arrow[rr, bend left] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[r, bend left] & \bullet[d] \arrow[r] & \cdots
\end{tikzcd}$$

Composition of 3 arrows is 0.

Abelian quotients of triangulated categories

Theorem (Buan, Marsh, Reiten 2006)

Let Λ be a hereditary algebra. Let $\mathscr{C} = D^b(\Lambda)/\tau^{-1}[1]$ (the cluster category). If T is a cluster-tilting (i.e maximally rigid) object, then $\mathscr{C}/\tau T \cong \operatorname{mod} \operatorname{End}_{\mathscr{C}}(T)$

Theorem (König, Zhu 2007)

Let $\mathscr{C}$ be a triangulated category. Let $\mathscr{X}$ be a maximally rigid subcategory. Then $\mathscr{C}/\mathscr{X}$ is an abelian category.

Theorem (Grimeland, J. 2015)

Let $\mathscr{C}$ be a triangulated category, and let $T \in \mathscr{C}$. Then $\operatorname{Hom}_{\mathscr{C}}(T,-)$ is a full and dense (i.e quotient) functor if and only if:

- **a** *If $T_1 \to T_2$ is a right min. morphism in $\operatorname{Add} T$, then any triangle $T_1 \to T_2 \to X \xrightarrow{h} \Sigma T_1$ satisfies $\operatorname{Hom}_{\mathscr{C}}(T,h) = 0$.*
- **b** *For any T-supported $X \in \mathscr{C}$ we can find a triangle as above with $T_1, T_2 \in \operatorname{Add} T$ and $\operatorname{Hom}_{\mathscr{C}}(T,h) = 0$.*

d-abelian quotients of $d+2$-angulated categories

- $\mathscr{C}$: a k-linear, Hom-finite, $d+2$-angulated category with split idempotents, d-suspension Σ^d and Serre functor S.
- T: An object in $\mathscr{C}$ with endomorphism algebra Γ
- $\mathscr{D}$: The essential image of $\operatorname{Hom}_{\mathscr{C}}(T,-) : \mathscr{C} \to \operatorname{mod}\Gamma$.

Theorem (J., Jørgensen; arxiv:1712:07851)

$\mathscr{D}$ is d-cluster-tilting in $\operatorname{mod}\Gamma$ and $\operatorname{Hom}_{\mathscr{C}}(T,-)$ is full iff the following conditions are all satisfied:

a *Suppose that $M \in \operatorname{mod}\Gamma$ satisfies $\operatorname{Ext}^j_\Gamma(\mathscr{D}, M) = 0$ for $1 \leqslant j \leqslant d-1$, and that $T_1 \xrightarrow{f} T_0$ is a morphism in $\operatorname{Add} T$ for which*

$$Hom_{\mathscr{C}}(T, T_1) \xrightarrow{Hom_{\mathscr{C}}(T,f)} Hom_{\mathscr{C}}(T, T_0) \to M \to 0$$

is a minimal projective presentation in $\operatorname{mod}\Gamma$. Then there exists a completion of f to a $(d+2)$-angle in $\mathscr{T}$,

$$T_1 \xrightarrow{f} T_0 \xrightarrow{h_{d+1}} X_d \xrightarrow{h_d} \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} \Sigma^d T_1,$$ *which satisfies $Hom_{\mathscr{C}}(T, h_d) = 0$.*

a* *Suppose that $N \in \operatorname{mod}\Gamma$ satisfies $\operatorname{Ext}^j_\Gamma(N, \mathscr{D}) = 0$ for $1 \leqslant j \leqslant d-1$, and that $ST_1 \xrightarrow{g} ST_0$ is a morphism in $\operatorname{Add} ST$ for which*

$$0 \to N \to Hom_{\mathscr{C}}(T, ST_1) \xrightarrow{Hom_{\mathscr{C}}(T,g)} Hom_{\mathscr{C}}(T, ST_0)$$

is a minimal injective copresentation in $\operatorname{mod}\Gamma$. Then there exists a completion of g to a $(d+2)$-angle in $\mathscr{T}$,

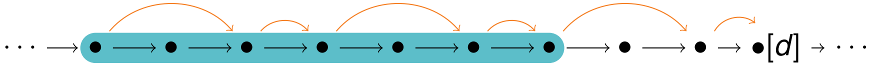
$$\Sigma^{-d} ST_0 \xrightarrow{h_{d+1}} X_d \xrightarrow{h_d} \cdots \xrightarrow{h_2} X_1 \xrightarrow{h_1} ST_1 \xrightarrow{g} ST_0,$$ *which satisfies $Hom_{\mathscr{C}}(T, h_2) = 0$.*

b *Suppose that $X \in \mathscr{C}$ is indecomposable and satisfies $Hom_{\mathscr{C}}(T, X) \neq 0$. Then there exists a $(d+2)$-angle in $\mathscr{T}$,*

$$T_d \to \cdots \to T_0 \to X \xrightarrow{h} \Sigma^d T_d,$$

with $T_i \in \operatorname{Add} T$ for $0 \leqslant i \leqslant d$, which satisfies $Hom_{\mathscr{C}}(T, h) = 0$.

Example



Look at the same example as before: $k\mathbb{A}_7/\langle\text{paths of length 3}\rangle$
The objects satisfying **a, a*** and **b**
Regain the original category: