# TRUNCATED PATH ALGEBRAS, <br> A GEOMETRIC AND HOMOLOGICAL STEPPING STONE 

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 Babson, Bleher, Chinburg, Goodearl, Shipman, Thomas Collaborators on the homological results: Dugas, Learned, Saorín$\Lambda=K Q / I$ is a finite dim'l algebra over a field $K=\bar{K}$; its Jacobson radical is $J$, and $L+1$ is its Loewy length, i.e., $L$ is minimal with $J^{L+1}=0$.

To $\Lambda$ we associate the following truncated path algebra

$$
\Lambda_{\text {trunc }}=K Q /\langle\text { all paths of length } L+1\rangle
$$

NOTE: - There is a surjective algebra homomorphism

$$
\Lambda_{\text {trunc }} \rightarrow \Lambda,
$$

and $\Lambda_{\text {trunc }}$ is the only truncated path algebra with quiver $Q$ and Loewy length $L+1$ that affords such a surjection.

- Clearly $\Lambda$-mod is embedded in $\Lambda_{\text {trunc }}-\bmod$ as a full subcategory. Moreover, for any dimension vector $\mathbf{d}$ of $Q$, the classical affine variety $\boldsymbol{R e p}_{\mathbf{d}}(\Lambda)$ (parametrizing the isomorphism classes of $\mathbf{d}$-dimensional $\Lambda$-modules) is embedded in $\boldsymbol{R e p}_{\boldsymbol{d}}\left(\Lambda_{\text {trunc }}\right)$ as a closed subvariety.

Observe that the finite dimensional basic hereditary algebras play a comparable role relative to the algebras with acyclic Gabriel quivers.

## Motivations

- Truncated algebras sport a very interesting representation theory, far more complex than that of hereditary algebras, but still significantly more accessible than that of algebras with arbitrary relations.

In studying homological and geometric aspects of $\Lambda$-mod, it has turned out helpful to move back and forth between $\Lambda$ and $\Lambda_{\text {trunc }}$. E.g.:

- The irreducible components of the varieties $\operatorname{Rep}_{\mathbf{d}}(\Lambda)$ are irreducible subvarieties of the $\boldsymbol{\operatorname { R e p }}_{\mathbf{d}}\left(\Lambda_{\text {trunc }}\right)$, and hence are contained in components of $\operatorname{Rep}_{\mathbf{d}}\left(\Lambda_{\text {trunc }}\right)$.
- Given any $\Lambda$-module $M$, the degenerations of $M$ over $\Lambda$ coincide with the degenerations of $M$ over $\Lambda_{\text {trunc }}$.

GOAL: Bring the representation theory of truncated path algebras up to the level attained for hereditary algebras.

Today: Primary focus on homological features, secondary on geometric properties.

## A. The homology of Truncated path ALGEBRAS

If $\Lambda$ is a truncated path algebra, then:

- All syzygies in $\Lambda$-mod are direct sums of principal left ideals.
- The global and finitistic dimensions of $\Lambda$ are understood (theoretically and computationally). In particular, the left and right little finitistic dimensions coincide with the big and are readily obtainable from $Q$ and $L$.
- To recognize the modules of finite projective dimension, one need not even compute syzygies - there is a structural criterion that singles them out (almost) "on sight".

In the following, I will bypass the basic homological attributes of truncated path algebras and focus on their tilting behavior.

## A.1. Tilting for general $\Lambda$

Miyashita's duality for arbitrary finite dim'l $\Lambda$.
Let $\mathcal{P}^{<\infty}(\Lambda$-mod) be the full subcategory of $\Lambda$-mod consisting of the modules of finite projective dimension. Clearly, this is a resolving subcategory of $\Lambda$-mod, i.e., it contains all projectives and is closed under extensions and kernels of surjective homomorphisms. Moreover, for any $M \in \Lambda$-mod, the category

$$
{ }^{\perp}\left({ }_{\Lambda} M\right)=\left\{X \in \Lambda-\bmod \mid \operatorname{Ext}_{\Lambda}^{i}(X, M)=0 \forall i \geq 1\right\}
$$

is resolving, whence so is the intersection

$$
\mathcal{P}^{<\infty}(\Lambda \text {-mod }) \cap{ }^{\perp}\left({ }_{\Lambda} M\right) .
$$

THM. [Miyashita] Whenever ${ }_{\Lambda} T_{\widetilde{\Lambda}}$ is a tilting bimodule, the functors $\operatorname{Hom}_{\Lambda}(-, T)$ and $\operatorname{Hom}_{\tilde{\Lambda}}(-, T)$ induce inverse dualities
$\mathcal{P}^{<\infty}(\Lambda-\bmod ) \cap{ }^{\perp}\left({ }_{\Lambda} T\right) \longleftrightarrow \mathcal{P}^{<\infty}(\bmod -\widetilde{\Lambda}) \cap{ }^{\perp}\left(T_{\widetilde{\Lambda}}\right)$.

## Broader perspective, still for arbitrary finite dim'l $\Lambda$.

THM. Let $\Lambda, \Lambda^{\prime}$ be finite dim'l algebras, and suppose that $\mathcal{C} \subseteq \mathcal{P}^{<\infty}(\Lambda-\bmod )$ and $\mathcal{C}^{\prime} \subseteq \mathcal{P}^{<\infty}\left(\bmod -\Lambda^{\prime}\right)$ are resolving subcategories of $\Lambda$-mod and $\bmod -\Lambda^{\prime}$, resp.
If $\mathcal{C}$ is dual to $\mathcal{C}^{\prime}$ by way of contravariant functors

$$
F: \mathcal{C} \rightarrow \mathcal{C}^{\prime} \text { and } F^{\prime}: \mathcal{C}^{\prime} \rightarrow \mathcal{C}
$$

then there exists a tilting bimodule ${ }_{\Lambda} T_{\Lambda^{\prime}}$ with the following properties:

- $\left.F \cong \operatorname{Hom}_{\Lambda}(-, T)\right|_{\mathcal{C}}$ and $\left.F^{\prime} \cong \operatorname{Hom}_{\Lambda^{\prime}}(-, T)\right|_{\mathcal{C}^{\prime}}$
- $\mathcal{C}^{\prime} \subseteq{ }^{\perp}\left(T_{\Lambda^{\prime}}\right) \quad$ and $\quad \mathcal{C} \subseteq{ }^{\perp}\left({ }_{\Lambda} T\right)$.

In particular, $\exists$ duality $\quad \mathcal{P}^{<\infty}(\Lambda$-mod $) \longleftrightarrow \mathcal{C}^{\prime} \quad$ if and only if the tilting module ${ }_{\Lambda} T$ as guaranteed by the theorem is Ext-injective relative to the objects of $\mathcal{P}^{<\infty}(\Lambda-\bmod )$, and $\mathcal{C}^{\prime}=\mathcal{P}^{<\infty}\left(\bmod -\Lambda^{\prime}\right) \cap^{\perp}\left(T_{\Lambda^{\prime}}\right)$. THUS: Any duality $\mathcal{P}^{<\infty}(\Lambda-\bmod ) \longleftrightarrow \mathcal{P}^{<\infty}\left(\bmod -\Lambda^{\prime}\right)$ is induced by a tilting bimodule which is two-sided Extinjective relative to the modules of finite projdim.

This fact puts a spotlight on a concept which was introduced by Auslander and Reiten, namely that of a strong tilting module. I will not present Auslander and Reiten's original definition, but instead give a characterization which can readily be seen to be equivalent.

DEF. [Auslander-Reiten] A tilting module $T \in \Lambda$-mod is strong in case $T$ is
relatively Ext-injective in $\mathcal{P}^{<\infty}(\Lambda$-mod $)$,
i.e., $\mathcal{P}^{<\infty}(\Lambda-\bmod ) \cap^{\perp}\left({ }_{\Lambda} T\right)=\mathcal{P}^{<\infty}(\Lambda-\bmod )$.

THM. [Auslander-Reiten] $\Lambda$-mod contains a strong tilting module if and only if the category $\mathcal{P}^{<\infty}(\Lambda$-mod) is contravariantly finite in $\Lambda$-mod.
In the positive case, there is a unique basic strong tilting module $T \in \Lambda$-mod, namely the direct sum of the indecomposable relatively Ext-injective objects of $\mathcal{P}^{<\infty}(\Lambda-\bmod )$.

## A.2. Strongly tilting truncated path algebras

In this section, $\Lambda$ is a truncated path algebra with quiver $Q$ and Loewy length $L+1$. In this setting, the theory that governs strong tilting is in place.

THM I. The category $\mathcal{P}^{<\infty}(\Lambda$-mod $)$ is contravariantly finite, and the minimal $\mathcal{P}^{<\infty}(\Lambda$-mod $)$-approximations of the simple modules are known personally.
Moreover, there is an explicit description of the basic strong tilting module ${ }_{\Lambda} T$. In particular, $T$ is constructible from $\underset{\sim}{\alpha}$ and $\underset{\sim}{L}$. The corresponding strongly tilted algebra $\widetilde{\Lambda}=K \widetilde{Q} / \widetilde{I}$ can in turn be determined from these data.

The homology of $\Lambda$ is governed by the following subdivision of the primitive idempotents $e_{1}, \ldots, e_{n}$ of $\Lambda$ : $e_{i}$ is called precyclic if $e_{i}$ is the source of a path which ends on an oriented cycle. The attribute postcyclic is dual, and $e_{i}$ is critical if $e_{i}$ is both pre- and postcyclic.

Primitive idempotents of $\Lambda$ versus those of $\widetilde{\Lambda}$ :

- Since $K_{0}(\Lambda) \cong K_{0}(\widetilde{\Lambda})$, the quiver $\widetilde{Q}$ has the same number of vertices as $Q$, say $\widetilde{e}_{1}, \ldots, \widetilde{e}_{n}$. It turns out that there is a canonical correspondence between the vertices of $Q$ and those of $\widetilde{Q}$. In sequencing the $\widetilde{e}_{i}$, I will assume that the order of the lineup reflects this correspondence. This makes the following unambiguous: An idempotent $\widetilde{e}_{i}$ is a critical vertex of $\widetilde{Q}$ if $e_{i}$ is critical in $Q$. (Caveat: These concepts do not pertain to the quiver $\widetilde{Q}$. The latter quiver teems with oriented cycles in general.)
DEF. • The idempotent of $\widetilde{\Lambda}$ which plays the key role in the homological behavior of mod $-\widetilde{\Lambda}$ is

$$
\widetilde{\mu}=\sum_{\text {critical }} \widetilde{e}_{i} .
$$

- The critical core of $\widetilde{M} \in \bmod -\widetilde{\Lambda}$ is the unique largest subfactor $V / U$ of $\widetilde{M}$ such that $\operatorname{top}(V / U) \widetilde{\mu}=\operatorname{top}(V / U)$ and $\operatorname{soc}(V / U) \widetilde{\mu}=\operatorname{soc}(V / U)$.
Here "largest" means "of highest dimension".

The simple left $\Lambda$-modules of finite projective dimension are those which correspond to the non-precyclic vertices of $Q$. By contrast, the simple right $\widetilde{\Lambda}$-modules $\widetilde{S}_{i}=\widetilde{e}_{i} \widetilde{\Lambda} / \widetilde{e}_{i} \widetilde{J}$ of finite projective dimension are those that correspond to the non-critical vertices of $\widetilde{Q}$ :

THM II. proj $\operatorname{dim} \widetilde{S}_{i}<\infty$ iff $\widetilde{e}_{i}$ is non-critical.
It is, in fact, completely understood what the right $\widetilde{\Lambda}$ modules of finite projective dimension look like.

THM III. For $\widetilde{M} \in \operatorname{Mod}-\widetilde{\Lambda}$, the following are equivalent:

- proj $\operatorname{dim} \widetilde{M}_{\widetilde{\Lambda}}<\infty$.
- The critical core of $\widetilde{M}$ is a direct sum of copies of the critical cores of the $\widetilde{e}_{i} \widetilde{\Lambda}$ (personally available).
$\operatorname{EXPL} . \Lambda=K Q /\langle$ all paths of length 3$\rangle$, where $Q$ is


Clearly, $e_{1}, e_{2}$ are the only critical vertices of $Q$. The basic strong tilting module is $T=\bigoplus_{i=1}^{6} T_{i}$ :

$\widetilde{\Lambda}=\operatorname{End}_{\Lambda}(T)^{\text {op }}=$ strong tilt of $\Lambda$. Quiver of $\operatorname{End}_{\Lambda}(T)$ :


The indecomposable projective right $\widetilde{\Lambda}$-modules $\widetilde{e}_{i} \widetilde{\Lambda}$ :


Question: The obvious next question is this: Is the tilting module $T_{\widetilde{\Lambda}}$ strong also in mod- $\widetilde{\Lambda}$ ?

Answer: NO, in general. The answer is positive precisely when all precyclic vertices of $Q$ are also postcyclic, meaning that $Q$ does not have a precyclic source.

Followup Question: Does mod- $\widetilde{\Lambda}$ have its own strong tilting module, i.e., is $\mathcal{P}^{<\infty}(\bmod -\widetilde{\Lambda})$ always contravariantly finite, even when $T_{\widetilde{\Lambda}}$ fails to be a strong tilting module?

Here the answer is an unqualified YES.
THM IV. The category $\mathcal{P}^{<\infty}(\bmod -\widetilde{\Lambda})$ is always contravariantly finite in $\bmod -\widetilde{\Lambda}$.
Moreover, one can pin down the minimal $\mathcal{P}^{<\infty}(\bmod -\widetilde{\Lambda})-$ approximations of the $\widetilde{S}_{i}$ and the basic strong tilting module $\widetilde{T} \in \bmod -\widetilde{\Lambda}$ from $\widetilde{Q}$ and $\widetilde{I}$. (There is a theoretical description which allows for construction of these modules.)

So how does this game continue?
Let $\widetilde{\Lambda}=\operatorname{End}_{\widetilde{\Lambda}}(\widetilde{T})$. Is the tilting bimodule $\widetilde{\widetilde{\Lambda}} \widetilde{T}_{\widetilde{\Lambda}}$ strong on both sides? The answer provides the strongest evidence so far for my assertion that the transit from $\Lambda$ to $\widetilde{\Lambda}$ effectively symmetrizes the original truncated path algebra from a homological viewpoint.

THM V. YES.
CONSEQUENCE: $\widetilde{\widetilde{\Lambda}} \cong \widetilde{\widetilde{\Lambda}}$.

## B. The geometry of truncated path ALGEBRAS

Let $\Lambda$ be a truncated path algebra with quiver $Q$ and Loewy length $L+1$, and let $\mathbf{d}$ be a dimension vector.

The irreducible components of the varieties $\operatorname{Rep}_{\mathbf{d}}(\Lambda)$ are classifiable, based on quiver and Loewy length of $\Lambda$.
The most crucial invariant in this classification is the radical layering

$$
\mathbb{S}(M)=\left(J^{l} M / J^{l+1} M\right)_{0 \leq l \leq L} \text { for } M \in \Lambda-\bmod
$$

For any semisimple sequence ( $=$ a sequence of semisimples in $\Lambda$-mod), say
$\mathbb{S}=\left(\mathbb{S}_{0}, \ldots, \mathbb{S}_{L}\right)$ with $\underline{\operatorname{dim}} \mathbb{S}=\mathbf{d}$, we let $\operatorname{Rep} \mathbb{S}$ be the subvariety of $\operatorname{Rep}_{\mathrm{d}}(\Lambda)$ consisting of the points that represent modules with radical layering $\mathbb{S}$.

- Then all $\operatorname{Rep} \mathbb{S}$ are irreducible subvarieties of $\operatorname{Rep}_{d}(\Lambda)$, and the irred components are among the closures $\overline{\operatorname{Rep} \mathbb{S}}$.

To sift the radical layerings $\mathbb{S}$ for which $\overline{\operatorname{Rep} \mathbb{S}}$ is an irreducible component out of the pool of all semisimple sequences $\mathbb{S}$ with dimension vector $\mathbf{d}$, an additional module invariant needs to be considered: Namely,
$\Gamma(M):=$ the number of realizable semisimple sequences that govern some filtration of $M$.
This invariant is in turn upper semicontinuous on $\Lambda$-mod and thus generically constant on irreducible subvarieties of $\operatorname{Rep}_{\mathbf{d}}(\Lambda)$.

SIFTING THM. $\overline{\operatorname{Rep} \mathbb{S}}$ is an irreducible component of $\operatorname{Rep}_{\mathrm{d}}(\Lambda)$ iff there exists a module $M$ with radical layering $\mathbb{S}$ such that $\Gamma(M)=1$.

The geometric structure of the irreducible components of the varieties $\operatorname{Rep}_{\mathrm{d}}(\Lambda)$ is comparatively simplistic.

THM. Let $\Lambda$ be truncated and $\overline{\operatorname{Rep} \mathbb{S}}$ an irreducible component of $\operatorname{Rep}_{\boldsymbol{d}}(\Lambda)$. Then $\operatorname{Rep} \mathbb{S}$ has a finite open cover, each patch of which is isomorphic to a full affine space. In particular, the variety $\operatorname{Rep} \mathbb{S}$ is smooth and its closure $\overline{\operatorname{Rep} \mathbb{S}}$ is rational.

As for the generic structure of the modules in a given irreducible component $\overline{\operatorname{Rep} \mathbb{S}}:$

THM. Let $\Lambda$ and $\mathbb{S}$ be as in the preceding theorem. Then a generic minimal projective presentation of the modules in $\overline{\operatorname{Rep} \mathbb{S}}$ may be pinned down "on sight" from the quiver and Loewy length of $\Lambda$.

Readily available: The generic socle layering of the modules in $\overline{\operatorname{Rep} \mathbb{S}}$, as well as the generic $K$-dimension of the endomorphism rings. Not under control in general: The generic values of the dimension vectors of the direct summands arising in indecomposable decompositions of the modules in a given component.

EXPL. Let $\Lambda=\mathbb{C} Q /\langle$ the paths of length 3$\rangle$, where $Q$ is the quiver

and $\mathbf{d}=(1,1, \ldots, 1) \in \mathbb{N}^{\top}$.
$\boldsymbol{R e p}_{\mathbf{d}}(\Lambda)$ has 28 irreducible components, 6 of which are given in terms of these generic modules:


