Hasse-Witt matrices and period integrals

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1. Collaborators

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2. Introduction

- We are trying to build a bridge between the *B-model* of mirror symmetry, and arithmetic geometry. This program was inspired by works of Candelas, de la Ossa and Rodriguez-Villegas in 2000, where such striking connections have been observed in an important case via direct computations. Special cases also appeared in works of Dwork, Katz, C.D. Yu, etc.
- In the B-model, the central objects of study are *period integrals*, in particular their Taylor series expansions at the *large complex structure limit* (LCSL) point.
- In arithmetic geometry, we are interested in counting the number of points of an algebraic variety over a finite field.

3. An example

- $f = a_1 x_1^2 + a_0 x_1 x_2 + a_2 x_2^2$: a Calabi-Yau hypersurface in $\mathbb{P}^1$: i.e. a Kahler manifold with $c_1 = 0$.
- Suppose the coefficients a_0, a_1, a_2 live in the finite field $\mathbb{F}_p$, and we compute the number of points N_p of the hypersurface over $\mathbb{F}_p$.
- $N_p = 1 + (\frac{\Delta}{p})$, where ( ) is the Legendre symbol.

4. An example

- Next we regard the coefficients in $f = a_1 x_1^2 + a_0 x_1 x_2 + a_2 x_2^2$ to be complex numbers. f is a global section of the anticanonical line bundle over $\mathbb{CP}^1$. For generic f the zero loci $V(f)$ consists of two points on the Riemann sphere.
- Period integrals for Calabi-Yau hypersurface: integrals of holomorphic top form over cycles.
- By Leray-Poincare residue, the unique period integral of the hypersurface

$$I = \int_{\gamma_0} \frac{x_1 dx_2 - x_2 dx_1}{f} = \Delta^{-\frac{1}{2}}$$

  where γ_0 is the unique generator of $H_1(\mathbb{CP}^1 - V(f))$ normalized such that the constant term of I is 1, and $\Delta = a_0^2 - 4a_1 a_2$.

5. **An example**

- In mirror symmetry, a particular degenerate anticanonical section called the *large complex structure limit* LCSL of of special interest, near which the mirror map is defined.
- In our case, the LCSL is $s_0 = x_1 x_2$, i.e. $a_0 = 1, a_1 = a_2 = 0$. For $\mathbb{CP}^{n-1}$ a LCSL is given by $s_0 = x_1 \ldots x_n$. In general LCSL is characterized by the property that the period sheaf has maximal unipotent monodromy at the point.
- Let $P = P(\frac{a_1}{a_0}, \frac{a_2}{a_0})$ denote the Taylor series of $a_0 I$ at the LCSL, then one checks that
$$N_p - 1 = \Delta^{\frac{p-1}{2}} = ({}^{(p-1)}P) a_0^{p-1} (mod\ p)$$
where ${}^{(p-1)}P$ denotes the truncation of P up to degree $p-1$ in $1/a_0$.
- Thus The analytic period at LCSL and point counting over $\mathbb{F}_p$ mod p for almost all p determine each other.
- **Remark**: Thinking of f as living in the universal family of Calabi-Yau hypersurface in $\mathbb{P}^1$ parametrized by a_0, a_1, a_2, the local behavior of the analytic period at the LCSL determines point counting mod p everywhere/globally in the parameter space.

6. Hasse-Witt and Periods

We prove that the above relation holds for a large class of hypersurfaces.

- Let $X = X^n$ be a toric variety or flag variety G/P of dimension n defined over $\mathbb{Z}$. Consider the universal family of CY hypersurfaces in X, given by the complete linear system of global sections of the anticanonical line bundle.
- **Remark**: The result can be extended to CY or general type complete intersections.
- Let Y be a smooth hypersurface in the family, taking reduction mod p, Fulton's fixed point formula implies $1 + (-1)^{n-1} HW_p = N_p (mod\ p)$, where HW_p is the Hasse-Witt invariant that records the (matrix of) the action of the Frobenius operator: $H^{n-1}(Y, \mathcal{O}_Y) \to H^{n-1}(Y, \mathcal{O}_Y)$.
- Let s_0 denote the large complex structure limit (LCSL) in the toric case given by union of toric divisors, or the candidate LCSL [H-Lian-Zhu'13] in the $X = G/P$ case given by union of codim=1 strata of the *projected Richardson stratification*: e.g. when $X = G(2,4)$, $s_0 = x_{12}x_{23}x_{34}x_{41}$, where x_{ij} are Plücker coordinates.

7. Main theorem relating Hasse-Witt and periods

- Extend s_0 to a basis of $\Gamma(X, K_X^{-1})$, and let $a_0, \ldots, a_N$ denote the dual basis. Let I denote the unique holomorphic period under the canonical global normalization of the holomorphic top form given by a global Poincare residue formula [Lian-Yau'11] at s_0, scaled such that the constant term equals 1. Let $P = P(a_1/a_0, \ldots, a_N/a_0)$ denote the Taylor series of $a_0 I$ at the LCSL, and ${}^{(p-1)}P$ denotes the truncation of P up to degree $p-1$ in $1/a_0$.
- **Theorem** [H-Lian-Yau-Yu'18] $HW_p = ({}^{(p-1)}P)a_0^{p-1}(mod\ p)$.
- **Remark**: The result is independent of the choice of extending s_0 to a basis.

8. Global normalization of the holomorphic top form

- Lian-Yau gave a global normalization of the holomorphic top form on the hypersurface, given by
$$Res \frac{\Omega}{f}$$
where Ω is a holomorphic n-form on certain principal bundle over X, such that Ω/f descends to a rational form on X with pole along the hypersurface $V(f)$. Taking residue then gives rise to a holomorphic top form on the hypersurface.
- For example, when $X = \mathbb{P}^n$, $\Omega = \sum_{k=0}^{n} (-1)^k x_k dx_0 \wedge ... \wedge \hat{dx_k} \wedge ... \wedge dx_n$.

9. Idea of proof

- Proof is based on
- **Lemma**: if on a local affine chart, $f = g(t)(dt_1 \wedge \dots \wedge dt_n)^{-1}$, then HW_p is equal to the coefficient of $(t_1 \dots t_n)^{p-1}$ in the local expansion of $g(t)^{p-1}$.
- The lemma relies on the compatibility of Grothendieck duality with Cartier operator.
- Let X be toric, and $f = \sum_I a_I x^I$. Take the affine torus chart $X - V(s_0)$. The above lemma implies that
$(1/\mathrm{a}_0^{p-1}) HW_p =$
$1 + \sum_{k=1}^{p-1} \sum_{k_1 u_{I_1} + \cdots + k_l u_{I_l} = 0, \sum k_j = k, I_j \neq 0} \binom{p-1}{k_1, k_2, \cdots, k_l, p-1-k} (\frac{a_{I_1}}{a_0})^{k_1} \cdots (\frac{a_{I_l}}{a_0})^{k_l}$
where $k = k_1 + \dots + k_l$.

10. Idea of proof

- On the other hand, the unique analytic period integral at the LCSL

$$I = \frac{1}{(2\pi\sqrt{-1})^n} \int_\gamma \frac{dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n f(t)}$$

along the cycle γ: $|t_1| = |t_2| = \cdots |t_n| = 1$, where $f(t)$ denotes f/s_0 written in terms of the torus t coordinates. So I equals the coefficient of the constant term in the Laurent expansion of $f(t)^{-1}$:

$$I = \frac{1}{a_0}(1+\sum_{k=1}^{\infty}(-1)^k \sum_{k_1 u_{I_1}+\cdots+k_l u_{I_l}=0, \sum k_j = k, I_j \neq 0} \binom{k}{k_1, k_2, \cdots, k_l} (\frac{a_{I_1}}{a_0})^{k_1} \cdots (\frac{a_{I_l}}{a_0})^{k_l})$$

- The congruence relation

$$\binom{p-1}{k_1, k_2, \cdots, k_l, p-1-k} \equiv (-1)^k \binom{k}{k_1, k_2, \cdots, k_l} \mod p$$

implies our result.

11. A few corollaries

- There is a version of the result for general type hypersurfaces.
- **Corollary** [H-Lian-Yau-Yu'18] The Hasse-Witt matrix for a generic smooth toric hypersurface is invertible.
- This corollary is needed to discuss the p-adic version of the result. When $X = \mathbb{P}^n$, it was proved by Adolphson.
- The proof is an induction on the size of the toric polytope.
- **Remark**: From the above local algorithm for HW_p applied to the torus chart, one can verify directly that HW_p satisfies a certain linear PDE system τ called the *tautological system* mod p. On the other hand, [H-Lian-Zhu'13] has proved that this τ is equivalent to the Gauss-Manin connection for period integrals. This generalizes an old result of Igusa-Manin-Katz that HW_p solves the Picard-Fuchs equation mod p.
- It is clear that the combinatorial structure of the LCSL plays an important role in the proof. It may be worthwhile to investigate this on a more conceptual level, to further "demystify" the LCSL.

12. **Idea of proof: the $X = G/P$ case**

- For the case $X = G/P$, one uses the Bott-Samelson-Demazure-Hansen resolution of Schubert varieties to construct a torus chart on $X - V(s_0)$, on which $s_0 = t_1 ... t_n (dt_1 \wedge ... \wedge dt_n)^{-1}$, where $t_1, ..., t_n$ are coordinates on the torus.
- In addition, it is a resolution of a rational singularity, which allows us to use differential forms with poles to compute HW_p.
- The proof then goes similar to the toric case.

13. **Example of** $G(2,4)$

- Let X be Grassmannian $G(2,4)$. Then $X = G/P$ with $G = SL(4)$ and
$$P = \{\begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix}\}.$$
- The Weyl group is $W = S_4$ and $W_P = S_2 \times S_2$. A shortest representative of the longest element in W/W_P: $w_P = (13)(24) = (23)(34)(12)(23)$.
- The Bott-Samelson-Demazure-Hansen resolution of the Schubert variety in G/B corresponding to w_P: $Z_{w_P} = P_1 \times P_2 \times P_3 \times P_4/B^4$ with
$$P_1 = \{\begin{pmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & t_1 & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix}\},\ P_2 = \{\begin{pmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & t_2 & \star \end{pmatrix}\},$$
$$P_3 = \{\begin{pmatrix} \star & \star & \star & \star \\ t_3 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix}\} \text{ and } P_4 = \{\begin{pmatrix} \star & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & t_4 & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix}\}.$$

14. Example of $G(2,4)$

- The largest Schubert cell is $\{\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\}P/P$ with coordinates (a,b,c,d).
- The affine coordinate $(t_1,\cdots,t_4)\in\mathbb{A}^4$ on a chart on Z_{w_P}:

$$\{[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & t_2 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 & 0 & 0 \\ t_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t_4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}]\}.$$

- So we have a map $\psi\colon Z_{w_P}\to X$ under this local chart on the torus $t_1t_2t_3t_4\neq 0$ given by

$$a=\frac{1}{t_1t_3},b=-\frac{t_1+t_4}{t_1t_2t_3t_4},c=\frac{1}{t_1},d=-\frac{1}{t_1t_2}$$

- ψ restricts to an isomorphism on the torus $t_1t_2t_3t_4\neq 0$.

15. Example of $G(2,4)$

- Let $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$ be the basis of any two plane. The Plücker coordinates x_{ij} are the determinants of i, j columns. The section $s_0 = x_{12}x_{23}x_{34}x_{14}$.
- We have $s_0 = -ad(ad - bc)(da \wedge db \wedge dc \wedge dd)^{-1}$. A direct calculation shows that $\psi^* s_0 = t_1 t_2 t_3 t_4 (dt_1 dt_2 dt_3 dt_4)^{-1}$. The other sections of $H^0(X, L)$ can also be written as homogenous polynomials of x_{ij} of degree 4, which in turn can be expressed in terms of the torus coordinates.

16. 2nd Main theorem: p-adic version of the result

- Now let a_I be p-adic integers. Let $g(a_I) := \frac{P(a_I)}{P(a_I^p)}$ as a power series. Then g satisfies Dwork congruences

$$g(a_I) \equiv \frac{{}^{(p^s-1)}(P(a_I))}{{}^{(p^{s-1}-1)}(P)((a_I)^p)} \mod p^s$$

- **Theorem** [H-Lian-Yau-Yu'18] Let $\hat{a}_I = \lim_{s\to\infty} a_I^{p^s}$, then $g(\hat{a}_I)$ gives the *unit root* of the zeta function of Y_f (after reduction mod p). In addition, the algorithm is effective.
- **Remark**: For example, for elliptic curves, this unit root gives complete information of the local zeta function.
- **Theorem** [H-Lian-Yau-Yu'18] Similar results hold for general type hypersurfaces in a toric variety.
- For the case of $\mathbb{P}^n$, this was a recent conjecture of Vlasenko.
- A slightly weaker version of the result generalizes to $X = G/P$.

17. Remarks about the proof

- The proof adopts a method of Katz regarding the formal expansion map of Crystalline cohomology, in the case with log poles.
- In the case with log poles, we do not have exact understanding of the kernel of this formal expansion map. A trick is used to get around this trouble. We also need a convergence result proved by Vlasenko.

18. Concluding remarks

- This work is a first step in our attempt to construct the p-adic B-model.
- The result implies that the fundamental period at the LCSL and the counting of rational points mod p for almost all p determine each other. In particular, the local information of this period at LCSL determines the point counting mod p everywhere on the parameter space.
- The next step is to relate the periods with monodromy at the LCSL with arithmetic of the hypersurface. The hope is that counting points determines all the periods at the LCSL. The work of Candelas et al in 2000 gave strong hints in this direction. We expect implications in both arithmetic geometry and mirror symmetry: in mirror symmetry, the point counting shall imply strong relations of periods at different LCSL.